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An Elliptic Boundary Value Problem with Unbounded Coefficients in a Half Space

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Equazioni a derivate parziali. — *An Elliptic Boundary Value Problem with Unbounded Coefficients in a Half Space.* Nota di ANTONIO BOVE (*) (***), BRUNO FRANCHI (*) (***) e ENRICO OBRECHT (**) (***), presentata (****) dal Corrisp. G. CIMMINO.

RIASSUNTO. — In questa nota diamo alcuni risultati su di una classe di problemi al contorno per equazioni ellittiche a coefficienti polinomiali in un semispazio. Si stabilisce l'esistenza di una parametrice destra e di una parametrice sinistra del problema; si stabiliscono inoltre stime a priori del problema e di quello aggiunto.

1. Elliptic boundary value problems in unbounded domains have not yet been completely studied, at least when either the boundary or the coefficients are unbounded. Contributions to such a kind of problems were given by several authors (see, e.g., [1], [8], [9], [10], [12]).

In this note, we state a result in this direction, when the domain is a half space and the principal symbol of the differential operator is an elliptic polynomial in the couple of variables (x, ξ).

A problem of this type arises in the study of boundary value problems for parabolic equations with mixed lateral conditions (see [3]) and contains, as a particular case, boundary value problems in a half space for the Hamiltonian operator of the quantum mechanical anharmonic oscillator.

Proofs of results announced here will appear elsewhere [4]. More precisely, in [4] the boundary value problem depends on a complex parameter q; this dependence is necessary for applications to parabolic problems. For sake of simplicity, here we shall everywhere omit the dependence on the complex parameter q.

In what follows, undefined notations will be as in [5].

2. Let $b \in \mathrm{N}$, $m \in \mathbf{R}$. A C^∞–function $p(x, \xi)$ will be called a symbol of order m if, for every multiindices α, β,

$$|\mathrm{D}_x^\alpha \mathrm{D}_\xi^\beta p(x, \xi)| \leq \mathrm{C}_{\alpha,\beta} (1 + |x|^b + |\xi|)^{m - |\alpha|/b - |\beta|}, \quad \forall (x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n.$$

In the following, we shall denote by $\mathrm{S}^{(b,m)}$ this class of symbols.

It turns out that $\mathrm{S}^{(b,m)}$ is a subclass of Beals' [2] and that it is possible to develop in a standard way a calculus for pseudodifferential operators

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associated to symbols of $S^{(b,m)}$. Furthermore, analogously to Hörmander's [7] case, this class contains parametrices of its "elliptic" operators (see also [11]).

Moreover, we can associate—in a natural way—with this class of symbols a scale of Sobolev spaces with weight; more precisely, if $s \geq 0$, we shall denote by $B^{(b,s)}(\mathbf{R}^n)$ the domain of

$$(-\Delta + |x|^{2b} + 1)^{s/2}$$

in $L^2(\mathbf{R}^n)$, with the graph norm.

It turns out that $u \in B^{(b,s)}(\mathbf{R}^n)$ if and only if $u \in H^s(\mathbf{R}^n)$, $|x|^{bs} u \in L^2(\mathbf{R}^n)$. These spaces have properties quite analogous to those of Sobolev spaces in bounded domains; in particular, $B^{(b,s)}(\mathbf{R}^n)$ is compactly embedded in $B^{(b,s')}(\mathbf{R}^n)$ when $s' < s$.

To this regard, we remark that a fundamental role in the proof of these results is played by the interpolation property of these spaces.

By what we have already said, the functions in these spaces have traces on the hyperplane $x_n = 0$ in the usual sense. More precisely, denoting by γ_k the k-th order trace operator, we have the following

THEOREM 1. *If $s > k + \frac{1}{2}$, γ_k is surjective and continuous from $B^{(b,s)}(\mathbf{R}^n)$ onto $B^{(b,s-k-1/2)}(\mathbf{R}^{n-1})$ and possesses a bounded right inverse.*

3. In the following we shall always denote by P a differential operator such as

$$P(x, D) = \sum_{|\alpha+\beta| \leq 2m} a_{\alpha\beta}\, x^{b\alpha} D^{\beta}, \tag{1}$$

where $a_{\alpha\beta}$ are complex numbers.

Moreover P will satisfy the following "ellipticity" conditions:

A_1) P is properly elliptic;

A_2) $\displaystyle\sum_{|\alpha+\beta| = 2m} a_{\alpha\beta}\, x^{b\alpha} \xi^{\beta} \neq 0 \quad , \quad \forall (x, \xi) \in \mathbf{R}^{2n} \setminus \{0\}$.

In addition to this, we shall consider a matrix of boundary differential operators $B = \| B_{jk} \|$ $(j = 1, \cdots, m\,;\, k = 0, \cdots, 2m-1)$, where

$$B_{jk}(x', D') = \sum_{|\gamma+\delta| \leq \mu_j - k} b^{jk}_{\gamma\delta}\, x'^{b\gamma} D^{\delta}. \tag{2}$$

Here μ_j is a suitable non-negative integer and $b^{jk}_{\gamma\delta}$ are complex numbers. We shall also suppose that

B) The operators B_{jk} cover P (Šapiro-Lopatinskiĭ condition).

This condition may be formulated in terms of Calderón-Hörmander's projector. Let us now say a few words about this operator.

If $u \in \mathcal{S}(\overline{\mathbf{R}^n_+})$, let us denote by $\gamma u = (\gamma_0 u, \cdots, \gamma_{2m-1} u)$ the $2m$-vector whose components are the Cauchy data of u, and by u^0 the function u extended by zero to all of $\mathbf{R}^n$.

If $u \in \mathcal{S}(\overline{\mathbf{R}^n_+})$ we define

$$\tilde{P}(\gamma u) = P(u^0) - (Pu)^0 ;$$

then, for any $v \in (\mathcal{S}(\mathbf{R}^{n-1}))^{2m}$, we put

$$Qv = \gamma((E\tilde{P}v)/\mathbf{R}^n_+),$$

where E denotes a parametrix of the operator P.

It can be shown that Q is a matrix-valued pseudodifferential operator in our classes (i.e. a matrix with entries belonging to our classes) and that its principal symbol $q(x', \xi') : \mathbf{C}^{2m} \to \mathbf{C}^{2m}$ is the projection operator onto the Cauchy data of solutions of the ordinary differential equation

$$P(x', 0, \xi', D_n) u(x', \xi', x_n) = 0$$

which are bounded on the positive half axis.

It is shown in [6] that condition B) is equivalent to

B′) The principal symbol of $(I - Q) \oplus B$ is injective from $\mathbf{C}^{2m}$ into $\mathbf{C}^{2m} \oplus \mathbf{C}^m$,

which, in turn, in view of hypothesis A_1), is equivalent to

B″) The principal symbol of BQ is surjective from $\mathbf{C}^{2m}$ onto $\mathbf{C}^m$.

Now let us suppose $s \geq 2m$; given

$$f \in B^{(b, s-2m)}(\mathbf{R}^n_+), \; g \in \mathop{\times}_{j=1}^{m} B^{(b, s-\mu_j-1/2)}(\mathbf{R}^{n-1}),$$

we consider the following boundary value problem

$$\text{(3)} \qquad \begin{cases} Pu = f & \text{in } \mathbf{R}^n_+, \\ B(\gamma u) = g & \text{in } \mathbf{R}^{n-1}. \end{cases}$$

Our main result is the following one:

THEOREM 2. *If hypotheses* A_1), A_2), B) *are satisfied, then the operator* $P \oplus B\gamma$ *is continuous from* $B^{(b,s)}(\mathbf{R}^n_+)$ *into* $B^{(b, s-2m)}(\mathbf{R}^n_+) \times (\mathop{\times}_{j=1}^{m} B^{(b, s-\mu_j-1/2)}(\mathbf{R}^{n-1}))$ *and has a right and a left parametrix.*

This fact implies that the a-priori estimate

$$\| u ; B^{(b,s)}(\mathbf{R}^n_+) \| \leq C \Big(\| f ; B^{(b, s-2m)}(\mathbf{R}^n_+) \| + \\ + \sum_{j=1}^{m} \| g_j ; B^{(b, s-\mu_j-1/2)}(\mathbf{R}^{n-1}) \| + \| u ; L^2(\mathbf{R}^n_+) \| \Big)$$

holds true toghether with its dual estimate. Therefore problem (3) *has finite dimensional kernel and cokernel.*

We explicitly note that the boundary conditions in problem (3) need not be normal.

4. We give an example of application of the preceding result. Let

$$P(x, D) = -\Delta + |x|^2 \tag{4}$$

be the Hamiltonian operator of the quantum mechanical harmonic oscillator and

$$B(x', D) = \sum_{j=1}^{n} \alpha_j D_j + \sum_{j=1}^{n-1} \beta_j x_j + \gamma$$

a boundary differential operator of type (2). Here $\alpha_j, \beta_j, \gamma$ are complex numbers. Then $B(x', D)$ covers $P(x, D)$ if and only if

$$\sum_{j=1}^{n-1} \alpha_j \xi_j + \sum_{j=1}^{n-1} \beta_j x_j + i\alpha_n \left(\sum_{j=1}^{n-1} (x_j^2 + \xi_j^2)\right)^{1/2} + \gamma \neq 0$$

for every $(\xi_1, \cdots, \xi_{n-1}) \in \mathbf{R}^{n-1} \setminus \{0\}$. In particular, Dirichlet and Neumann problems for the operator (4) have finite dimensional kernel and cokernel.

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