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On partial stability and boundedness of degree k

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Equazioni differenziali ordinarie. — *On partial stability and boundedness of degree k.* Nota di OLUSOLA AKINYELE, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dà le definizioni di parziale stabilità e limitatezza di grado k, e dà condizioni sufficienti perchè un sistema differenziale abbia queste proprietà.

§ 1. INTRODUCTION

Lyapunov [4] posed the problem of the stability of motion with respect to a part of the variables, otherwise known as partial stability. Ever since several authors [1, 2, 3, 5, 6, 8] have validated the possibility of applying the theorems of Lyapunov's second method including their modifications and generalizations for this specific problem.

In this paper, by employing Lyapunov—like functions and the theory of systems of differential inequalities, we develop a new comparison theorem and introduce new concepts of stability and boundedness of solutions of a differential system in Euclidean spaces, with respect to a part of the variable We then investigate sufficient conditions for such stability and boundedness properties to hold.

§ 2. PRELIMINARIES AND DEFINITIONS

We consider a system of differential equations

$$(1) \qquad \frac{d\omega}{dt} = f(t, \omega) \quad , \quad f(t, 0) = 0 \quad , \quad \omega(t_0) = \omega_0$$

where $f \in C(R^+ \times R^n, R^n)$. Here R^+ denotes the non-negative real line, R^n the Euclidean space and $\|\cdot\|$ any convenient norm.

Let $\omega_1, \omega_2, \cdots, \omega_m$ $(m > 0)$ be a part of the variables $(\omega_1, \omega_2, \cdots, \omega_n)$ and $n = m + p$, $p \geq 0$. Let $x_i = \omega_i$ $(i = 1, 2, \cdots, m)$ and the rest of the variables be $y_i = \omega_{m+j}$, $(j = 1, 2, \cdots, n - m = p)$. Then

$$\omega = (x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_p)$$

and

$$\|x\| = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2} \quad , \quad \|y\| = \left(\sum_{j=1}^{p} y_j^2\right)^{1/2}$$

(*) Nella seduta del 16 dicembre 1978.

and

$$\| \omega \| = \left(\sum_{i=1}^{n} \omega_i^2 \right)^{1/2} = (\| x \|^2 + \| y \|^2)^{1/2} .$$

Define for $\rho > 0$, $S_\rho = \{\omega \in R^n : \| x \| < \rho , 0 \leq \| y \| < \infty , t \geq 0\}$. We assume that the solutions of (1) are y-component extendable, that is, any solution $\omega(t)$ is defined for all $t \geq 0$ for which $\| x \| < \rho$. We also assume conditions on f which guarantee existence and uniqueness of solutions of (1).

DEFINITION 2.1. Denote by $\mathscr{D}$, the set of all continuous functions ϕ defined on R^+ which are monotonically increasing and differentiable in R^+ and such that

$$\phi(t) \geq 1 \quad \text{for} \quad 0 \leq t < \infty \qquad \text{and} \quad \lim_{t \to \infty} \phi(t) = b \geq 1$$

where b is a real number.

The purpose of this paper is to develop the partial stability and boundedness properties of degree k with respect to ϕ for the system (1). These properties are now defined.

DEFINITION 2.2. The trivial solution $\omega = 0$ of (1) is said to be: PN_1: equistable of degree k with respect to ϕ relative to $\omega_1, \omega_2, \cdots, \omega_m$ or partially equistable of degree k with respect to ϕ if given $\varepsilon > 0$, $t_0 \in R^+$ there exist $\delta = \delta(\varepsilon, t_0)$ and $\phi \in \mathscr{D}$ such that every solution $\omega(t, t_0, \omega_0)$ of (1) satisfies,

$$\| \phi^k(t) \, x(t, t_0, \omega_0) \| < \varepsilon , \qquad \text{for} \quad t \geq t_0$$

provided $\| \phi^k(t_0) \, \omega_0 \| < \delta$.

PN_2: partially uniformly stable of degree k with respect to ϕ if δ in (i) is independent of t_0.

Corresponding to the definitions (S_1)–(S_8) of [3] we can formulate (PN_1)–(PN_8).

Remark. If $\phi(t) \equiv 1$, then our definitions (PN_1)–(PN_8) reduce to the definitions (P_1)–(P_8) of [3, § 3.11], that is, partial stability notions of system (1). If $k = 1$, our definitions reduce to the concepts of partial stability with respect to the function ϕ, which are also new. If $m = n$ and $\phi(t) \equiv 1$, then our definitions (PN_1)–(PN_8) reduce to the definitions (S_1)–(S_8) of [3].

Analogous to the definitions (PN_1)–(PN_8) we may formulate definitions of partial boundedness of solutions of (1) of degree k with respect to ϕ on the basis of definition 2.2 and corresponding definitions (B_1)–(B_8) of [3, § 3.13]. These considerations are straightforward and the reader may formulate them by himself.

Corresponding to the given differential system (1), we shall consider the scalar differential equation

$$\text{(2)} \qquad \frac{du}{dt} = g(t, u) \quad , \quad u(t_0) = u_0 > 0$$

where $g \in C(R^+ \times R^+, R)$.

§ 3. MAIN RESULTS

We now state and prove a variant of the standard comparison theorem which is the main tool of this paper.

THEOREM 3.1 (*k^{th} degree Comparison Theorem*). *Assume that*

(i) $V \in C(R^+ \times R^n, R^+)$, $V(t, x)$ *is locally Lipschitzian in* x *for each* $t \in R^+$;

(ii) $g \in C(R^+ \times R^+, R)$, $g(t, u)$ *is nondecreasing in* u *for each* $t \in R^+$, *and the maximal solution* $r(t, t_0, r_0)$ *of the scalar differential equation* (2) *exists to the right of* t_0;

(iii) $\phi(t)$ *is a continuous function which is monotonically increasing and differentiable in the interval* $[0, \infty)$ *and the conditions*

$$\phi(t) \geq 1 \quad , \quad 0 \leq t < \infty \quad , \quad \lim_{t \to \infty} \phi(t) = c \geq 1$$

hold.

(iv) *For* $k \geq 1$ *and* $(t, \phi^k(t)\, x(t)) \in R^+ \times R^n$,

$$D^+ V(t, \phi^k(t)\, x(t)) \leq g(t, V(t, V(t, \phi^k(t)\, x(x)))$$

Then if $\omega(t, t_0, \omega_0)$ *is any solution of* (1) *existing for* $t \geq t_0$, *such that* $V(t_0, \phi^k(t_0)\, \omega_0) \leq r_0$, *then*

$$V(t, \phi^k(t)\, \omega(t, t_0, \omega_0)) \leq r(t, t_0, r_0) \quad , \quad t \geq t_0 .$$

Proof. For $t \geq t_0$, define

$$m(t) = V(t, \phi^k(t)\, \omega(t)) ,$$

then

$$\begin{aligned} m(t+h) - m(t) &= V(t+h, \phi^k(t+h)\, \omega(t+h)) - V(t, \phi^k(t)\, \omega(t)) \\ &= V(t+h, \phi^k(t+h)\, \omega(t+h)) - \\ &\quad - V(t+h, \phi^k(t+h)\ \omega(t) + \phi^k(t+h)\, hf(t, \omega)) \\ &\quad + V(t+h, \phi^k(t+h)\, \omega(t) + \phi^k(t+h)\, hf(t, \omega)) \\ &\quad - V(t, \phi^k(t)\, \omega(t)) . \end{aligned}$$

$$\begin{aligned} &\leq L(t+h) \, \| \phi^k(t+h)\, \omega(t+h) - \phi^k(t+h)\, \omega(t) - \phi^k(t+h)\, hf(t, \omega) \| \\ &\quad + V(t+h, \phi^k(t+h)\, \omega(t) + \phi^k(t+h)\, hf(t, \omega)) \\ &\quad - V(t, \phi^k(t)\, \omega(t)) . \end{aligned}$$

$$\frac{m(t+h)-m(t)}{h} \leq \mathrm{L}(t+h)\,\phi^k(t+h)\left\|\frac{\omega(t+h)-\omega(t)}{h}-f(t,\omega)\right\|$$

$$+\frac{1}{h}\,[\mathrm{V}(t+h\,,\phi^k(t+h)\,[\omega(t)+hf(t\,,\omega)])-$$

$$-\mathrm{V}(t\,,\phi^k(t)\,\omega(t))]$$

Using (iii)

$$\mathrm{D}^+ m(t) \leq \mathrm{D}^+ \mathrm{V}(t\,,\phi^k(t)\,\omega(t)) \leq g(t\,,m(t)) \quad , \qquad t \geq t_0$$

An application of Theorem 1.4.1 of [3] therefore implies

$$\mathrm{V}(t\,,\phi^k(t)\,\omega(t\,,t_0\,,\omega_0)) \leq r(t\,,t_0\,,r_0) \qquad \text{for} \quad t \geq t_0\,.$$

Remark. Our result obviously contains the well-known comparison theorem. All that is required to verify this is to set $\phi(t) \equiv 1$ for all t in Theorem 3.1. This result thus generalises the usual comparison one.

We now state another variant of our comparison result.

THEOREM 3.2. *Suppose the hypothesis of Theorem* 3.1 *hold except that instead of (iv) we assume that for* $k \geq 1$.

$$\mathrm{D}^+ \mathrm{V}(t\,,\phi^k(t)\,\omega(t)) + \Phi(\|\phi^k(t)\,\omega(t)\|) \leq g(t\,,\mathrm{V}(t\,,\phi^k(t)\,\omega(t)))\,,$$

for $(t\,,\phi^k(t)\,\omega(t)) \in \mathrm{R}^+ \times \mathrm{R}^n$, *where* $\Phi(u) \geq 0$ *is continuous for* $u \geq 0$, $\Phi(0) = 0$ *and* $\Phi(u)$ *is strictly increasing in* u. *Assume in addition that* $g(t\,,u)$ *is nondecreasing in* u *for each* $t \in \mathrm{R}^+$. *Then* $\mathrm{V}(t_0\,,\phi^k(t_0)\,\omega_0) \leq r_0$ *implies that*

$$\mathrm{V}(t\,,\phi^k(t)\,\omega(t)) + \int_{t_0}^{t} \Phi(\|\phi^k(s)\,\omega(s)\|)\,\mathrm{d}s \leq r(t\,,t_0\,,r_0)$$

for $t \geq t_0$.

Proof. Define

$$m(t) = \mathrm{V}(t\,,\phi^k(t))\,\omega(t)) + \int_{t_0}^{t} \Phi(\|\phi^k(s)\,\omega(s))\,\mathrm{d}s$$

and proceed as in the last theorem and the result follows.

We now use our comparison result to investigate sufficient conditions for the partial stability and boundedness of degree k, with respect to the function ϕ.

THEOREM 3.3. *Assume that there exist* $\mathrm{V}(t\,,\omega)$ *and* $g(t\,,u)$ *satisfying the following*:

(i) $g \in \mathrm{C}(\mathrm{R}^+ \times \mathrm{R}^+, \mathrm{R})\,, g(t\,,0) = 0$, *and* $g(t\,,u)$ *is nondecreasing in* u *for each* $t \in \mathrm{R}^+$,

(ii) $V \in C(R^+ \times S'_\rho, R^+)$, $V(t, o) = o$, $V(t, \omega)$ *is locally Lipschitzian in* ω, *and there exists* $a \in \mathcal{K} = \{\phi \in C([o, \rho), R), \phi(o) = o$ *and* $\phi(r)$ *is strictly monotone increasing in* $r\}$, *such that*

$$a(\|\phi^k(t)\, x\|) \leq V(t, \phi^k(t)\, \omega(t)) \qquad \text{for} \quad (t, \phi^k(t)\, \omega) \in R^+ \times S_\rho$$

where $S_\rho = \{\omega \in R^n : \|\omega\| < \rho\}$ *and* $\phi(t)$ *is the function defined in Theorem* 3.1 (iii).

(iii) *For* $k \geq 1$, *and* $(t, \phi^k(t)\, \omega(t)) \in R^+ \times S_\rho$,

$$D^+ V(t, \phi^k(t)\, \omega(t)) \leq g(t, V(t, \phi^k(t)\, \omega(t)))$$

where ϕ *is a continuous function as defined in Theorem* 3.1 (iii).

Then (i) *the equistability of the trivial solution of* (2) *implies the partial equistability of degree k with respect to the function* ϕ *of the trivial solution of* (1).

(ii) *the equi-asymptotic stability of the trivial solution of* (2) *implies the partial equi-asymptotic stability of degree k with respect to the function* ϕ *of the trivial solution of* (1).

Proof. On the basis of our comparison result and the standard arguments with obvious modifications the results follow [cfr. 3].

THEOREM 3.4. *Assume that hypothesis* (i), (ii) *and* (iii) *of the last theorem hold and that there exists* $b \in \mathcal{K}$ *such that*

$$V(t, \phi^k(t)\, \omega(t)) \leq b(\|\phi^k(t)\, \omega(t)\|) \quad , \quad (t, \phi^k(t)\, \omega(t)) \in R^+ \times S_\rho .$$

Then (i) *the equistability of the trivial solution of* (2) *implies the partial equi-stability of degree k with respect to the function* ϕ *of the trivial solution of* (1).

(ii) *the uniform stability of the trivial solution of* (2) *implies the partial uniform stability of degree k with respect to* ϕ *of the trivial solution of* (1).

Proof. The proof runs parallel to that of Theorem 3.3.

THEOREM 3.5. *Assume that hypothesis* (i), (ii) *and* (iii) *of Theorem* 3.3 *hold, and that there exist* $a, b \in \mathcal{K}$ *such that*

$$a(\|\phi^k(t)\, x\|) \leq V(t, \phi^k(t)\, \omega(t)) \leq b(\|x\| + \|y\|) .$$

Then (i) *the equi-asymptotic stability of the trivial solution of* (2) *implies the partial equi-asymptotic stability of degree k with respect to the function* ϕ *of the trivial solution of* (1).

(ii) *the uniform asymptotic stability of the trivial solution of* (2) *implies the partial uniform asymptotic stability of degree k with respect to the function* ϕ *of the trivial solution of* (1).

Proof. On the basis of the comparison result and routine standard arguments the proofs are straightforward.

Remark. If $k = 1$, then Theorems 3.3, 3.4 and 3.5 give sufficient conditions for the partial equistability, partial uniform stability and partial uniform asymptotic stability with respect to the function ϕ. If $\phi(t) \equiv 1$ and $p > 0$ then the various partial stability results in Theorems 3.3, 3.4 and 3.5 reduce to the known partial stability results [cfr. 3]. In the special case $\phi(t) \equiv 1$ and $p = 0$, Theorems 3.3, 3.4 and 3.5 reduce to the Lyapunov equistability, uniform stability and uniform asymptotic stability of the trivial solution $\omega = 0$ of the system (1). In the special case $\phi(t) = e^{t-t_0}$, $t \geq t_0$, $\beta > 0$, $\beta < \alpha$ and $g(t, V(t, e^{t-t_0}\omega(t)) = -\alpha V(t, e^{t-t_0}\omega(t))$, our result of Theoiem 3.5 is equivalent to the partial exponential asymptotic stability of the trivial solution of (1). For the definition of partial exponential asymptotic stability, the reader is referred to [7, § 1].

We can state and prove partial boundedness results (PB_1–PB_8) parallel to Theorem 3.3, 3.4 and 3.5 on the basis of our kth degree comparison result. However, these consideration are fairly straight-forward and so we omit details.

References

[1] C. Corduneanu (1971) – *Some problems concerning partial stability*, «Symp. Math.», *6*, Acad. Press, New York, 141–154.

[2] C. Corduneanu (1975) – *On partial stability for delay systems*, «Ann. Polon Maths.», *29*, 357–362.

[3] V. Lakshmikantham and S. Leela (1969) – *Differential and integral inequalities*, Vol. I, Acad. Press, New York.

[4] A. M. Lyapunov (1950) – *General problem of the Theory of Motion*, Moscow.

[5] A. S. Oziraner (1971) – *A contribution to the stability of motion relative to a part of the variables*. Vestnik Mosckov. Gos. Univ. Mat. Mekh, no. 1.

[6] A. S. Oziraner (1972) – *On asymptotic stability with respect to a part of the variables*. Vestnik. Moskov. Gos. Univ. Mat. Mekh., no. 1.

[7] A. S. Oziraner and V. V. Rumanstev (1972) – *The method of Lyapunov functions in the stability problem for motion with respect to a part of the varibles*, «P.M.M.», *36*, no. 2, 364–384.

[8] V. V. Rumianstev (1971) – *On asymptotic stability and istability of motion with respect to a part of the variables*, «P.M.M.», no. 1.