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Existence of Solutions Across Resonance in the Large for Semilinear Problems

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Analisi matematica. — *Existence of Solutions Across Resonance in the Large for Semilinear Problems.* Nota di P. J. McKENNA, presentata (*) dal Socio D. GRAFFI.

RIASSUNTO. — L'Autore considera l'equazione astratta:

$$\text{E}x + \lambda x = \text{N}x \tag{1}$$

con E operatore lineare, N operatore non lineare, λ parametro. Detti λ_0 e λ_1 due successivi autovalori di (1) (con $\text{N} = 0$), e sotto opportune condizione per N, dimostra che esiste un $\varepsilon > 0$, tale che per $\lambda_0 - \varepsilon < \lambda < \lambda_1$ la (1) ammette un insieme di soluzioni uniformemente limitate.

INTRODUCTION

The study of the existence of solutions across resonance was introduced by Cesari [1] where he studied the existence of solutions to equations of the form $\text{E}x + \alpha x = \text{N}x$, for small values of α, with suitable conditions on the linear operator E at resonance and the nonlinear operator N. Again in the framework of the alternative method, Mc Kenna [6, 7] and Cesari [2] showed that similar theorems could be proved for equations of the type $\text{E}x + \varepsilon \text{N}_1 x = \text{N}x$ for sufficiently small ε and suitable nonlinear N_1.

In this paper, we adopt a different approach, and show that in the presence of a now well understood geometric condition on N, the equation $\text{E}x + \alpha x = \text{N}x$ can be solved from as close to one eigenvalue as we desire to some point across the next eigenvalue.

THE MAIN RESULT

Let $\mathscr{H}$ be a Hilbert space, and let N be a continuous nonlinear bounded map from $\mathscr{H}$ to $\mathscr{H}$. We assume that E has a sequence of eigenvalues $\lambda_1 \leq \lambda_2, \cdots, \lambda_i \to +\infty$ with associated orthonormal eigenvectors ϕ_i.

If $\{\phi_i\}_{m+1}^{m+k}$ are the eigenvalues associated with eigenvalue zero, $\lambda_1 \leq \cdots \leq \lambda_m < 0 < \lambda_{m+k+1} \leq \cdots$, then we define a partial inverse K on the space of functions of the type

$$x = \sum_0^m c_i \phi_i + \sum_{m+k+1}^{\infty} c_i \phi_i \quad \text{and} \quad \text{K}x = \sum_{i=0}^m \frac{1}{\lambda_i} c_i \phi_i + \sum_{m+k+1}^{\infty} \frac{1}{\lambda_i} c_i \phi_i .$$

(*) Nella seduta dell'8 gennaio 1977.

If $I - P$ is the orthogonal projection onto these functions x, then $K(I - P)\mathscr{H} \to (I - P)\mathscr{H}$ is compact and since

$$(Kx , x) = \sum_{0}^{m} \frac{1}{\lambda_i} c_i^2 + \sum_{m+k+1}^{\infty} \frac{1}{\lambda_i} c_i^2 \qquad \text{so}$$

$$\frac{1}{\lambda_m} \| x \|^2 \leq (Kx , x) \leq \frac{1}{\lambda_{m+k+1}} \| x \|^2 . \tag{1}$$

We assume

$$\| Nx \| \leq M \quad \text{for all } x \in \mathscr{H} \tag{N$_1$}$$

$$\forall R_1 > 0 , \quad \exists R_2 > 0 \quad \text{and} \quad \delta : [0 , \infty) \to (0 , \infty) \tag{N$_2$}$$

such that if

$$x_0 \in P\mathscr{H} , \| x_0 \| \geq R_0 \qquad x_1 \in (I - P)\mathscr{H} , \| x_1 \| \leq R_1$$

then

$$(N(x_0 + x_1) , x_0) > \delta(\| x_0 \|) > 0 .$$

THEOREM I. *Under the foregoing general assumptions on* E *and the particular assumptions* N_1) *and* N_2) *on* N, *there exists* $\alpha_0 < 0$ *so that for every* $\alpha , \alpha_0 \leq \alpha < \lambda_{m+k+1}$, *the equation*

$$Ex - \alpha x = Nx \tag{2}$$

has at least one solution. Moreover for every $\alpha_1 , 0 \leq \alpha_1 < \lambda_{m+k+1}$ *there exists a uniformly bounded connected set of solutions for* $\alpha \in [\alpha_0 , \alpha_1]$.

Proof. We shall search for solutions (cfr. [2], [6] and [10]) of the coupled equation

$$0 = x - \{Px - K(I - P)Nx + \alpha K(I - P)x - PNx - \alpha Px\} = (I - T_\alpha)x . \tag{3}$$

We define a region Ω in $\mathscr{H}$ so that $d_{LS}(0 , I - T , \Omega)$ is equal to one. For any given $\alpha_1 , 0 < \alpha_1 < \lambda_{m+k+1}$, let

$$\Omega = \{x_0 + x_1 , x_0 \in P\mathscr{H} , x_1 \in (I - P)\mathscr{H} , \| x_0 \| \leq R_0 , \| x_1 \| \leq R_1 \|\}$$

where R_0 and R_1 are chosen so that

$$R_1 > 2(1 - \alpha_1/\lambda_{m+k+1})^{-1} \| K \| M , \tag{4}$$

where M is the constant in (N_2) and R_0 is then the corresponding constant in (N_3).

We shall determine below $\alpha_0 , \lambda_m < \alpha_0 < 0$, and show that for $\alpha \in [\alpha_0 , \alpha_1]$ $(I - \lambda T_\alpha) z \neq 0$ for $z \in \partial\Omega$ and $0 \leq \lambda \leq 1$.

a) Consider $z = x_0 + x_1 \| x_0 \| \leq R_0 , \| x_1 \| = R_1$. Then

$$((I - \lambda T_\alpha) z , x_1) = \| x_1 \|^2 - \lambda (K (I - P) N (x_0 + x_1) , x_1) - \lambda\alpha (K x_1 , x_1) .$$

In the case where $\alpha \leq 0$

$$((I - \lambda T_\alpha) z , x_1) \geq R_1^2 - \| K \| M R_1 + \alpha_0 \| K \| R_1^2$$
$$\geq R_1 \| K \| M + \alpha_0 \| K \| R_1^2 .$$

If $| \alpha_0 | < M/2 R_1$, then $((I - \lambda T_\alpha) z , x_1) > \delta$.

In the remaining case where $0 \leq \alpha \leq \alpha_1$ we have

$$((I - \lambda T_\alpha) z , x_1) \geq \| x_1 \|^2 - \| K \| M \| x_1 \lambda\alpha - \alpha_0/\lambda_{m+k+1} \| x_1 \|^2$$
$$\geq R_1^2 - \| K \| M R_1 - \alpha\lambda_{m+k+1}^{-1} R_1^2 \geq R_1 \| K \| M ,$$

the last inequality coming from (4).

Thus for α_0 sufficiently small, there exists $\delta > 0$ so that if $z = x_1 + x_1$, $\| x_0 \| \leq R_0 , \| x_1 \| = R_1$ then $((I - \lambda T_\alpha) z , x_1) \geq \delta$ for all λ_1 $0 \leq \lambda \leq 1$.

b) We now consider $z = x_0 + x_1 , \| x_0 \| = R_0 , \| x_1 \| \leq R_1$.
Then

$$((I - \lambda T_\alpha) z , x_0) = (1 - \lambda) \| x_0 \|^2 + \lambda (N (x_0 + x_1) , x_0) + \lambda\alpha \| x_0 \|^2 .$$

Since $(N (x_0 + x_1) , x_0) \geq \delta (\| x_0 \|) > 0$ on this part of the boundary, taking $\delta_1 = \delta (R_0)$ and $| \alpha_0 | < \delta_1/2 R_0^2$, we have $((I - \lambda T_\alpha) z , x_0) > \delta_2 > 0$ for all λ_1 $0 \leq \lambda \leq 1$.

Thus the equations $(I - T_\alpha) z = 0$ have solutions in Ω for all α, $\alpha_0 \leq \leq \alpha \leq \alpha_1$.

To establish the connectedness of a set of solutions, we need only quote the following Theorem, which is a slight variation of one found in [9].

THEOREM A. *Let* $F (t , x)$ *be a continuous compact map from* $[\alpha_0 , \alpha_1] \times \mathscr{H}$ *into* $\mathscr{H}$, *such that* $d_{LS} (I - F (t , x) , 0 , \Omega) = 1$ *for all* $t \in [\alpha_0 , \alpha_1]$, *and* $\| F (t , x) \| \geq \delta$ *on* $\partial\Omega$ *where* Ω *is a bounded open set of* $\mathscr{H}$. *Then there is a connected set of points* $\{(t , x) \mid t \in [\alpha_0 , \alpha_1] , x \in \Omega , F (t , z) = z\}$ *that meets both* $\{\alpha_0\} \times \overline{\Omega}$ *and* $\{\alpha_1\} \times \overline{\Omega}$.

Taking $F (t , z) = T_\alpha z$, it is clear that the theorem implies that there exists a connected set of solutions x_α to $(I - T_\alpha) x = 0$ for all $\alpha \in [\alpha_0 , \alpha_1]$. This concludes the proof of the theorem.

The reader will observe that in the proof of the theorem, we showed that for all $\alpha \in [\alpha_0 , \alpha_1]$ the inequality $\| (I - \lambda T_\alpha) z \| > \delta > 0$ held for all $z \in \partial\Omega$, $\lambda \in [0 , 1]$. This observation would allow us to include an additional nonlinear term εN_1 in the equation $Ex + \alpha x = Nx + \varepsilon N_1 (x)$, with the assumption that $N_1 : \mathscr{H} \to \mathscr{H}$ maps bounded sets into bounded sets. Then for $T_\alpha' = Px - K (I - P) (Nx + \varepsilon N_1 x) - \alpha K (I - P) x - P (Nx + \varepsilon N_1 x) - \alpha Px$ we would have $\| (I - \lambda T_\alpha') z \| \geq \delta/2$ and Theorem I would apply.

In the event of the reverse inequality (N_2') $(N(x_0 + x_1) x_0) \leq \delta < 0$ being satisfied instead of (N_2), a slight modification of the proof of Theorem I would yield.

THEOREM II. *Under the previous assumptions on* E *and the assumptions* (N_1) *and* (N_2') *on* N, *there exists* $\alpha_0 > 0$ *so that for every* α, $\lambda_m < \alpha < \alpha_0$ *the equations* $Lx - \alpha x = Nx$ *has at least one solution. Moreover, for every* α_1, $\lambda_m < \alpha_1 \leq 0$, *there exists a connected uniformly bounded set of solutions for* $\alpha \in [\alpha_1, \alpha_0]$.

If only the inequality (N_2'') $(N(x_0 + x_1), x_0) \geq 0$ is satisfied instead of $N_2)$ the following result holds.

THEOREM III. *Under the same general assumptions on* E *and assumptions* (N_1) (N_2'') *on* N, *then for every* $\alpha, 0 \leq \alpha < \lambda_{m+k+1}$, *the equation* $Ex - \alpha x = Nx$ *has at least a solution* $x_\alpha \in \mathscr{H}$. *Moreover, for every* α, $0 \leq \alpha \leq \alpha_1 < \lambda_{m+k+1}$ *the solutions* x_α *are uniformly bounded, and there exists a connected subset of the* x_α*'s for* $\alpha \in (0, \alpha_1)$.

Remarks. The connection between the geometric conditions N_2, N_2', N_2'' and the conditions of Landesman and Lazer [4], Lazer and Leach [5], Williams [10], and others is now well understood [3]. The observation that the Landesman and Lazer condition implies (N_2) we first made by Williams [10], and has been used extensively by Cesari [2], McKenna [6], and others.

In particular if $Ex = \dfrac{d^2 x}{dt^2} + m^2$ with periodic boundary conditions on $[0, 2\pi]$ and $\mathscr{H}$ is the space of $L^2[0, 2\pi]$, and $Nx = f(x) - h(t)$, then as Lazer and Leach [5], the condition N_2 is impled by

$$f(+\infty) = D \quad , \quad f(-\infty) = C$$

$$A = \frac{1}{2\pi}\int_0^{2\pi} h(t) \sin mt \, dt \qquad B = \frac{1}{2\pi}\int_0^{2\pi} h(t) \cos mt \, dt$$

and $2(D - C) > (A^2 + B^2)^{1/2}$.

In particular, if $\|h\| < D - C$, then condition (N_1) is satisfied uniformly at each eigenvalue $\lambda_i = i^2$ and thus all solutions of $+ x'' + m^2 x = g(x) + h(t)$, are bounded for $m^2 \in [0, R]$, with bound depending only on R.

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