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Operators of $A^{-}\Phi$ type

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Analisi funzionale. — *Operators of* A — Φ *type*. Nota (*) di NICOLAE TIȚA, presentata dal Socio G. SANSONE.

RIASSUNTO. — L'Autore generalizza una classe di operatori e dà una risposta negativa ad un problema posto da un altro autore.

1. Let E , F be normed spaces and $T : E \to F$ a linear and bounded operator ($T \in L(E, F)$). In the paper [3] the class of l_{A-p} operators is introduced in the following way

$$T \in l_{A-p} \quad \text{if} \quad \sum_i \left(\sum_j |a_{ij}| \, \alpha_j(T) \right)^p < \infty , \qquad 0 < p < \infty ,$$

where $A = \| a_{ij} \|$ is an infinite matrix satisfying some properties and $\alpha_j(T)$ are the approximations numbers ($\alpha_j(T) = \inf_k \| T - K \|$, $\dim K \leq j$).

In this paper, using the norm functions of R. Schatten [4], [6], the class of $l_{A-\Phi}$ operators is introduced. Also, replacing the numbers $\alpha_j(T)$ by the approximation numbers $\delta_j(T)$ and $d_j(T)$ [2], the classes $\tilde{l}_{A-\Phi}$ and $\bar{l}_{A-\Phi}$ are introduced.

2. Let Φ be a norm function of R. Schatten [4]

$$(\Phi : \hat{c} \to R_+) \quad ; \quad \Phi(x + y) \leq \Phi(x) + \Phi(y) , \qquad x , y \in \hat{c} ;$$

$$\Phi(\lambda x) = |\lambda| \cdot \Phi(x) , \qquad \lambda \in R , x \in \hat{c} \quad ; \quad \Phi(1, 0, 0, \cdots) = 1 ;$$

$$\Phi(x_1, x_2, \cdots, x_n, 0, 0, \cdots) = \Phi(|x_{i_1}|, |x_{i_2}|, \cdots, |x_{i_n}|, 0, 0, \cdots) ,$$

where $\hat{c}$ is the space of all sequences $x = \{x_1, x_2, \cdots, x_n, 0, 0, \cdots\}$, $x_i \in R$, $n < \infty$ and $i_1, i_2, \cdots, i_n$ is a permutation of $1, 2, \cdots, n$).

The approximation numbers $\delta_i(T)$ and $d_i(T)$ are defined in the following way

$$\delta_i(T) = \inf_{\mathscr{L}_n} \sup_{\|x\| \leq 1} \inf_{y \in \mathscr{L}_n} \| Tx - y \| \qquad \mathscr{L}_n \subset F , \dim \mathscr{L}_n \leq i , \qquad x \in E .$$

$d_i(T) = \inf_{E_i} \{ \| T |_{E_i} \| \}$, codim $E_i \leq i$ ($T|_{E_i}$ is the restriction of T to E_i).

DEFINITION 2.1. The operator $T \in l_{A-\Phi}(E, F)$ if

$$\Phi \left(\left\{ \sum_j a_{ij} \, \alpha_j(T) \right\}_i \right) < \infty \text{ (1)}.$$

(*) Pervenuta all'Accademia il 12 settembre. 1978.

(1) $a_{ij} \geq 0 ; a_{i0} \geq a_{i1} \geq a_{i2} \geq \cdots$. For $a_{ij} = \delta_{ij}$ we have the l_Φ class. $\Phi(x) = \sup_n \Phi(x_1, \cdots, x_n, 0, 0, \cdots)$ if $x \notin \hat{c}$.

If $\Phi(x) = \Phi_p(x) = \left(\sum_j |x_j|^p\right)^{1/p}$, $p \geq 1$ we get the class l_{A-p} [3]. Replacing the elements $\alpha_j(T)$ by $\delta_j(T)$ and $d_j(T)$ we get the classes $\tilde{l}_{A-\Phi}$ and $\bar{l}_{A-\Phi}$.

PROPOSITION 2.1. *$l_{A-\Phi}(E, F)$ is a linear quasinormed space with the quasinorm* $\|T\|_{A-\Phi} = \Phi\left(\left\{\sum_j a_{ij}\,\alpha_j(T)\right\}_i\right)$.

Proof. Consider the sequence

$$\{\alpha_0(T_1 + T_2), \alpha_1(T_1 + T_2), \alpha_2(T_1 + T_2), \cdots\}.$$

Since

$$\alpha_{2i+1}(T_1 + T_2) \leq \alpha_{2i}(T_1 + T_2) \leq \alpha_i(T_1) + \alpha_i(T_2), \qquad \forall i \in N,$$

we have

$$\sum_j a_{ij}\,\alpha_j(T_1 + T_2) \leq 2 \sum_j a_{ij}\,\alpha_{2j}(T_1 + T_2) \leq$$

$$\leq 2 \sum_j a_{ij}(\alpha_j(T_1) + \alpha_j(T_2)).$$

Hence

$$\|T_1 + T_2\|_{A-\Phi} = \Phi\left(\left\{\sum_j a_{ij}\,\alpha_j(T_1 + T_2)\right\}_i\right) \leq$$

$$\leq 2\,\Phi\left(\left\{\sum_j a_{ij}(\alpha_j(T_1) + \alpha_j(T_2))\right\}\right) \leq 2\,\Phi\left\{\left(\sum_i a_{ij}\,\alpha_j(T_1)\right) + \right.$$

$$\left. + \Phi\left(\sum_j a_{ij}\,\alpha_j(T_2)\right)\right\} = 2\,(\|T_1\|_{A-\Phi} + \|T\|_{A-\Phi}).$$

If $\lambda \in R$ we have

$$\|\lambda T\|_{A-\Phi} = \Phi\left(\left\{\sum_j a_{ij}\,\alpha_j(\lambda T)\right\}_i\right) = |\lambda| \cdot \Phi\left(\left\{\sum_j a_{ij}\,\alpha_j(T)\right\}\right) = |\lambda| \cdot \|T\|_{A-\Phi}.$$

Hence $l_{A-\Phi}(E, F)$ is a linear space and $\|T\|_{A-\Phi}$ is a quasinorm.

This proposition is also true for the classes $\tilde{l}_{A-\Phi}$ and $\bar{l}_{A-\Phi}$ since the properties of the elements $\delta_j(T)$ and $d_j(T)$ are similar to the properties of the elements $\alpha_j(T)$ [2].

REMARK 2.1. In the proof of Proposition 2.1 the condition, from [3], $|a_{i,2j}| + |a_{i,2j+1}| < M \cdot |a_{ij}|$ is not necessary. Hence the answer to the problem raised in [3] is negative.

Using the methods of [1] and [3] one proves:

PROPOSITION 2.2. *If* $T_1 \in L(E, F)$ *and* $T_2 \in l_{A-\Phi}(F, G)$ *then* $T_2 T_1 \in l_{A-\Phi}(E, G)$ *and* $\|T_2 T_1\|_{A-\Phi} \leq \|T_1\| \cdot \|T_2\|_{A-\Phi}$.

PROPOSITION 2.3. *If* F *is a Banach space then* $l_{A-\Phi}(E, F)$ *is complete.*

REMARK 2.2. If, in the definition of $l_{A-\Phi}$ class, the function $\psi_1(x) = \Sigma |x_i|$ is replaced by a function $\psi \neq \psi_1$ we get the class $l_{A-\Phi,\psi}$.

$$(T \in l_{A-\Phi,\psi} \quad \text{if} \quad \Phi(\{\psi(a_{ij}\,\alpha_j(T)\}) < \infty).$$

DEFINITION 2.2. Let Φ be a norm function. The conjugate function Φ^*_ψ is

$$\Phi^*_\alpha(x) = \sup_{y \in \hat{k}} \frac{\psi(x \cdot y)}{\Phi(y)}, \qquad \hat{k} = \{(x_i) \in \hat{c} \mid x_i \geq 0\}; \qquad x \cdot y = \{x_1 y_1, x_2 y_2, \cdots\}$$

$$(\Phi^*(x) = \sup_{y \in \hat{k}} \frac{\Sigma x_i y_i}{\Phi(y)} \quad ([4], [6]).$$

PROPOSITION 2.3. *Let* Φ, ψ (2) *be norm functions, then if* $T_1 \in l_{A-\Phi,\psi}(E, E)$ *and* $T_2 \in l_{\psi^*}(E, E)$, $T_1 T_2 \in l_{A-\Phi}(E, E)$.

Proof.

$$\Phi(\{\Sigma a_{ij}\,\alpha_j(T_1 \cdot T_2)\}) \leq 2\,\Phi\left(\left\{\sum_j a_{ij} \cdot \alpha_j(T_1) \cdot \alpha_j(T_2)\right\}_i\right) \leq$$

$$\leq 2\,\Phi(\{\psi(a_{ij}\,\alpha_j(T_1)) \cdot \psi^*(\alpha_j(T_2)\}) =$$

$$= 2\,\psi^*(\alpha_j(T_2)) \cdot \Phi(\{\psi(a_{ij}\,\alpha_j(T_1))\}) < \infty.$$

3. Between the classes $l_{A-\Phi}$, $\tilde{l}_{A-\Phi}$ and $\bar{l}_{A-\Phi}$ exist the following relations $l_{A-\Phi} \subset \tilde{l}_{A-\Phi}$ and $l_{A-\Phi} \subset \bar{l}_{A-\Phi}$, since $\delta_j(T) \leq \alpha_j(T)$ and $d_j(T) \leq \alpha_j(T)$ [2]. Also, if $T \in l_{A-\Phi}$ then $T' \in l_{A-\Phi}$ since $\alpha_j(T) = \alpha_j(T')$, where T′ is the conjugate operator of T.

REMARK 3.1. Using the expressions of the elements α_j, δ_j, d_j for the tensor product operator $T_1 \otimes T_2$ [5] we have that the classes $l_{A-\Phi}$, $\tilde{l}_{A-\varphi}$ and $\bar{l}_{A-\Phi}$ are stable with respect to the tensor product.

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(2) $\psi \neq \psi_1, \psi_\infty; \psi_\infty(x) = \sup_i |x_i|, x = \{x_1, x_2, \cdots, x_n, 0, 0, \cdots\}$.