Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Brian Fisher

Constant mappings and common fixed points

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **65** (1978), n.3-4, p. 104–106.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1978_8_65_3-4_104_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma bdim (Biblioteca Digitale Italiana di Matematica) SIMAI & UMI

http://www.bdim.eu/

Geometria differenziale. — *Constant mappings and common fixed points.* Nota di BRIAN FISHER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra che, se S e T sono applicazioni di uno spazio metrico X in sè tali che o

$$d(Sx, Ty) \leq bd(y, Sx) + cd(y, Ty), \qquad (0 \leq b, c < 1)$$

oppure

$$\{d(Sx, Ty)\}^2 \leq cd(y, Sx)\, d(y, Ty), \qquad (0 \leq c)$$

per tutti gli x, y di X, allora S e T hanno un unico punto fisso comune, z, ed inoltre $Sx = z$ per tutti gli x di X.

We first of all prove the following theorem:

THEOREM 1. *If* S *and* T *are mappings of the metric space* X *into itself satisfying the inequality*

$$d(Sx, Ty) \leq bd(y, Sx) + cd(y, Ty)$$

for all x, y *in* X, *where* $0 \leq b, c < 1$, *then* S *and* T *have a unique common fixed point* z. *Further,* S *is a constant mapping with* $Sx = z$ *for all* x *in* X.

Proof. If x is an arbitrary point in X, we have

$$d(Sx, TSx) \leq bd(Sx, Sx) + cd(Sx, TSx) = cd(Sx, TSx)$$

and, since $c < 1$, it follows that

$$TSx = Sx.$$

Thus the point $Sx = z$ is a fixed point of T. Further

$$d(Sz, z) = d(Sz, Tz) \leq bd(z, Sz) + cd(z, Tz) = bd(Sz, z)$$

and, since $b < 1$, it follows that

$$Sz = z.$$

Thus the point z is a common fixed point of S and T.

Now suppose that S and T have a second common fixed point w. Then

$$d(z, w) = d(Sz, Tw) \leq bd(w, Sz) + cd(w, Tw) = bd(z, w)$$

and, since $b < 1$, it follows that the common fixed point z must be unique.

(*) Nella seduta del 14 maggio 1977.

We now note that, since the point z is unique and since the arbitrary point x was mapped into z, S must map every point x in X into z. This completes the proof of the theorem.

Although we have proved that S is a constant mapping, T is not necessarily a constant mapping. To see this, let X be the set $\{0, 1, 3\}$ with metric

$$d(r, n) = |r - n| : \qquad r, n = 0, 1, 3 .$$

Define mappings S and T on X by

$$S(0) = S(1) = S(3) = T(0) = T(1) = 0, \qquad T(3) = 1 .$$

It is easily seen that the inequality of the theorem is satisfied with $b = c = \frac{1}{2}$, but T is not a constant mapping.

The inequality is also satisfied when $b = 0, c = \frac{1}{2}$ or when $b = \frac{1}{2}, c = 0$ and so T is not necessarily a constant mapping even if either $b = 0$ or $c = 0$.

In the particular case S = T, we of course have the following corollary:

COROLLARY. *If* T *is a mapping of the metric space* X *into itself satisfying the inequality*

$$d(Tx, Ty) \leq bd(y, Tx) + cd(y, Ty)$$

for all x, y *in* X, *where* $0 \leq b, c < 1$, *then* T *is a constant mapping.*

We now prove the following

THEOREM 2. *If* S *and* T *are mappings of the metric space* X *into itself satisfying the inequality*

$$\{d(Sx, Ty)\}^2 \leq cd(y, Sx)\, d(y, Ty)$$

for all x, y *in* X, *where* $0 \leq c$, *then* S *and* T *have a unique common fixed point* z. *Further,* S *is a constant mapping with* $Sx = z$ *for all* x *in* X.

Proof. If x is an arbitrary point in X, we have

$$\{d(Sx, TSx)\}^2 \leq cd(Sx, Sx)\, d(Sx, TSx) = 0$$

and it follows that

$$TSx = Sx .$$

Thus the point $Sx = z$ is a fixed point of T.

Further

$$\{d(Sz, z)\}^2 = \{d(Sz, Tz)\}^2 \leq cd(z, Sz)\, d(z, Tz) = 0 .$$

It follows that the point z is a common fixed point of S and T.

Now suppose that S and T have a second common fixed point w. Then

$$\{d(z, w)\}^2 = \{d(Sz, Tw)\}^2 \leq cd(w, Sz)\, d(w, Tw) = 0 .$$

It follows that the common fixed point z is unique and then that S maps every point x in X into z. This completes the proof of the theorem.

The example given above, with $c = 1$, again shows us that T is not necessarily a constant mapping.

In the particular case $S = T$, we have the following:

COROLLARY. *If* T *is a mapping of the metric space* X *into itself satisfying the inequality*

$$\{d(Tx, Ty)\}^2 \leq cd(y, Tx) d(y, Ty)$$

for all x, y *in* X, *where* $0 \leq c$, *then* T *is a constant mapping.*

We finally prove the following

THEOREM 3. *If* S *and* T *are mappings of the metric space* X *into itself satisfying the inequality*

$$d(Sx, Ty) < d(y, Sx) + d(y, Ty)$$

for all x, y *in* X, *with* $Sx \neq Ty$, *then* S *and* T *have a unique common fixed point* z. *Further,* S *is a constant mapping with* $Sx = z$ *for all* x *in* X.

Proof. Let x be an arbitrary point in X. Then if $Sx = TSx$, we have

$$d(Sx, TSx) < d(Sx, Sx) + d(Sx, TSx) = d(Sx, TSx),$$

giving a contradiction. It follows that the point $Sx = z$ is a fixed point of T.

Now suppose that $Sz \neq z$. Then

$$d(Sz, z) = d(Sz, Tz) < d(z, Sz) + d(z, Tz) = d(Sz, z),$$

giving a contradiction. It follows that the point z is a common fixed point of S and T.

If we now suppose that w is a second distinct common fixed point of S and T, then

$$d(z, w) = d(Sz, Tw) < d(w, Sz) + d(w, Tw) = d(z, w),$$

giving a contradiction. It follows that the common fixed point z is unique and then that S maps every point x in X into z. This completes the proof of the theorem.

In the particular case $S = T$, we have the following

COROLLARY. *If* T *is a mapping of the metric space* X *into itself satisfying the inequality*

$$d(Tx, Ty) < d(y, Tx) + d(y, Ty)$$

for all x, y *in* X, *with* $Tx \neq Ty$, *then* T *is a constant mapping.*