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Sufficient conditions for nonoscillation of forced n-th order retarded functional differential equations

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Equazioni differenziali ordinarie. — *Sufficient conditions for nonoscillation of forced n-th order retarded functional differential equations* (*). Nota (**) di LU–SAN CHEN e FONG–MING YU, presentata dal Socio G. SANSONE.

RIASSUNTO. — Per la equazione differenziale nonlineare con argomento ritardato

$$(r(t)\,x'(t))^{(n-1)} + \sum_{i=1}^{m} p_i(t)\,f_i(x[g_i(t)]) = q(t),$$

si danno condizioni sufficienti per r, p_i, f_i, g_i e q per le quali tutte le soluzioni non sono oscillatorie.

1. INTRODUCTION

The author [1] recently extended certain theorems of Graef and Spikes [3] and gave sufficient conditions for nonoscillation of n-th order nonlinear differential equations of the form

$$(r(t)\,x^{(n-1)}(t))' + p(t)\,g(x(t), x'(t), \cdots, x^{(n-1)}(t)) =$$
$$= h(t; x(t), x'(t), \cdots, x^{(n-1)}(t))$$

where $n \geqq 2$.

The purpose of this paper is to obtain sufficient conditions under which all bounded solutions of the retarded functional differential equation

$$(1) \qquad (r(t)\,x'(t))^{(n-1)} + \sum_{i=1}^{m} p_i(t)\,f_i(x[g_i(t)]) = q(t)$$

are nonoscillatory. For related results we refer the reader to the papers by Graef–Spikes [4], [5], Hammett [6], Londen [7] and Singh [8], [9].

Throughout this paper we assume the following conditions always hold:

(*a*) $r(t)$ *is continuous and eventually positive on* $[t_0, \infty)$, $t_0 \geqq 0$;

(*b*) $p_i(t)$, $i = 1, 2, \cdots, m$ *and* $q(t)$ *are continuous on* $[t_0, \infty)$;

(*c*) $g_i(t)$, $i = 1, 2, \cdots, m$ *are continuously differentiable nondecreasing functions on* $[t_0, \infty)$ *such that* $g_i(t) \leqq t$ *and*

$$\lim_{t\to\infty} g_i(t) = \infty$$

for $i = 1, 2, \cdots, m$;

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(*d*) $f_i(u)$, $i = 1, 2, \cdots, m$ *are continuous nondecreasing functions on* $(-\infty, \infty)$ *such that*

$$0 < \frac{f_i(u)}{u} \leq M$$

for $i = 1, 2, \cdots, m$ *and* $M > 0$.

A solution $x(t)$ of (1) is said to be continuable if it exists on some ray $[\alpha, \infty)$, $\alpha > 0$. A nontrivial solution of (1) is *oscillatory* if it is continuable and has arbitrarily large zeros. By a nonoscillatory solution we mean a continuable solution which is not oscillatory. The term " solution " for the remainder of this work will mean a nontrivial continuable solution.

2. MAIN RESULTS

THEOREM 1. *Assume that the equation* (1) *satisfy*

$$\int^{\infty} \frac{dt}{r(t)} < \infty, \tag{2}$$

$$\sum_{i=1}^{m} \int^{\infty} t^{n-2} p_i(t)\, dt < \infty, \tag{3}$$

and

$$\int^{\infty} t^{n-2} q(t)\, dt = \infty, \quad q(t) > 0 \quad on \quad [t_0, \infty). \tag{4}$$

Then all bounded solutions of (1) *are nonoscillatory.*

Proof. The techniques used in this proof are primarily adapted from those used in the work of Chen, Yeh and Lin, especially in [2]. Suppose to the contrary that $x(t)$ is an oscillatory bounded solution of (1). Due to oscillatory nature of $x(t)$, $(r(t)\,x'(t))^{(n-2)}$ must be oscillatory. In fact, if $(r(t)\,x'(t))^{(n-2)}$ in nonoscillatory, then $r(t)\,x'(t)$ assume one sign eventually. Since $r(t) > 0$, $x'(t)$ becomes nonoscillatory which in turn forces $x(t)$ to be nonoscillatory, a contradiction.

Hence $(r(t)\,x'(t))^{(n-2)}$ is oscillatory. Similarly $(r(t)\,x'(t))^{(n-3)}, \cdots, (r(t)\,x'(t))', x'(t)$ are all oscillatory.

Let $t_0 < \tau_{n-2} < \tau_{n-3} < \cdots < \tau_2 < \tau_1 < \tau_0$ be points such that

$$(r(\tau_j)\,x'(\tau_j))^{(j)} = 0 \qquad \text{for} \quad j = 0, 1, \cdots, n-2.$$

From (1)

$$r(t)\,x'(t) = -\int_{\tau_0}^{t}\int_{\tau_1}^{s_1}\cdots\int_{\tau_{n-2}}^{s_{n-2}} \sum_{i=1}^{m} p_i(y)\, x[g_i(y)]\, \frac{f_i(x[g_i(y)])}{x[g_i(y)]}\, dy\, ds_{n-2}\cdots ds_1 + \int_{\tau_0}^{t}\int_{\tau_1}^{s_1}\cdots\int_{\tau_{n-2}}^{s_{n-2}} q(y)\, dy\, ds_{n-2}\cdots ds_1 . \tag{5}$$

$$\geqq - M \int_{\tau_0}^{t} \int_{\tau_1}^{s_1} \cdots \int_{\tau_{n-2}}^{s_{n-2}} \sum_{i=1}^{m} p_i(y)\, x_+ [g_i(y)]\, dy\, ds_{n-2} \cdots ds_1$$

$$+ \int_{\tau_0}^{t} \int_{\tau_1}^{s_1} \cdots \int_{\tau_{n-2}}^{s_{n-2}} q(y)\, dy\, ds_{n-2} \cdots ds_1$$

$$= - M \int_{\tau_{n-2}}^{t} \frac{(y - \tau_{n-2})^{n-2}}{(n-2)!} \sum_{i=1}^{m} p_i(y)\, x_+ [g_i(y)]\, dy$$

$$+ \int_{\tau_0}^{t} \frac{(y - \tau_0)^{n-2}}{(n-2)!} q(y)\, dy \to \infty \qquad \text{as} \quad t \to \infty ,$$

since the first integral is bounded, where $x_+(t) = \max \{x(t), 0\}$. Thus $x'(t) > 0$ eventually and $x(t)$ is nonoscillatory, a contradiction. This contradiction proves our theorem.

THEOREM 2. *Assume that the equation* (1) *satisfy* (2) *and* (3). *If*

$$\int^{\infty} t^{n-2} q(t)\, dt = - \infty , \quad q(t) < 0 \qquad on \quad [t_0, \infty) . \tag{6}$$

Then all bounded solutions of (1) *are nonoscillatory.*

Proof. If $x(t)$ is an oscillatory bounded solution of (1), we put $z(t) = -x(t)$. Using $\int^{\infty} (-t^{n-2} q(t))\, dt = \infty$ and Theorem 1, we see our Theorem 2 holds.

THEOREM 3. *Assume that the equation* (1) *satisfy the following conditions*

$$\lim_{t \to \infty} \frac{t^{n-2}}{r(t)} = 0 , \tag{7}$$

$$\liminf_{t \to \infty} \frac{\int_{t_0}^{t} s^{n-2} q(s)\, ds}{r(t)} > 0 . \tag{8}$$

Then all nonoscillatory solutions of (1) *are eventually positive.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Suppose to the contrary that $x[g_i(t)] < 0$ for $i = 1, 2, \cdots, m$ and $t \geqq T > t_0$.

From (1)

$$(r(t)\, x'(t))^{(n-1)} \geqq q(t) \tag{9}$$

for $t \geqq T$. Hence there exists a constant k such that

$$r(t)\,x'(t) \geqq kt^{n-2} + \int_T^t \frac{(s-T)^{n-2}}{(n-2)!}\,q(s)\,ds\,. \tag{10}$$

It follows from (7), (8) and (10) that $x'(t)$ is bounded away from zero, which means that $x(t)$ is eventually positive, a contradiction. This contradiction proves our theorem.

THEOREM 4. *Assume that the equation* (1) *satisfy the condition* (7). *If*

$$\limsup_{t\to\infty} \frac{\int_{t_0}^t s^{n-2}\,q(s)\,ds}{r(t)} < 0\,, \tag{11}$$

then all nonoscillatory solutions of (1) *are eventually negative.*

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$$(a(t)\,x'(t))' + h(t, x(t), x'(t)) + q(t) f(x(t), x'(t)) = e(t, x(t), x'(t))$$

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