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Linear stochastic differential equations in Hilbert spaces, II

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Analisi matematica. — *Linear stochastic differential equations in Hilbert spaces*, II. Nota (*) di GIUSEPPE DA PRATO, MIMMO IANNELLI e LUCIANO TUBARO, presentata dal Socio G. SANSONE.

RIASSUNTO. — Proseguendo nello studio iniziato in [3] si danno risultati di esistenza per le equazioni differenziali stocastiche lineari in uno spazio di Hilbert.

Let $(\Omega, \mathscr{E}, P)$ be a probability space, $w = \{w_t, t \geq 0\}$ a Wiener process and $\mathscr{F} = \{\mathscr{F}_t, t \geq 0\}$ a family of σ-algebras non-anticipating with respect to the Wiener process (1).

Let H be a real Hilbert space, $A : D_A \subset H \to H$, $B : D_B \subset H \to H$ linear operators. Let us consider the following problem:

$$du = Au\,dt + Bu\,dw_t \qquad u(s) = u_0 . \tag{1}$$

If A and B are continuous operators which commute, it is easy to check that

$$u(t) = \exp\{(A - B^2/2)(t - s) + B(w_t - w_s)\}\, u_0 \tag{2}$$

is the unique solution of (1). We are led by (2) to consider the following assumptions

$H_1)_i$ B *is the infinitesimal generator of a* C_0*-group of type* (M_B, α_B) (2).

$H_1)_{ii}$ *It is* $D_A \subset D_{B^2}$, *and there exists a linear operator* $C : D_C \subset H \to H$ *which is the infinitesimal generator of a* C_0*-semi-group of type* (M_C, α_C) *and extends* $A - B^2/2$ (3).

Our aim is to define an evolution operator relative to problem (1) in order to give an explicit form for the solution of the non homogeneus problem:

$$du = (Au + f(t))\,dt + (Bu + g(t))\,dw_t \qquad u(s) = u_0 . \tag{3}$$

(*) Pervenuta all'Accademia il 26 luglio 1978.

(1) For the standard concepts of probability see, for example, [5].

(2) This means

$$|\exp tB| \leq M_B \exp(\alpha_B |t|) \qquad \forall t \in R.$$

(3) This means

$$|\exp tC| \leq M_C \exp(\alpha_C t) \qquad \forall t \geq 0 .$$

In section 1 we treat the commutative case in which it is possible to get explicit formulas; in the non commutative case (section 2) it is also possible to build an evolution operator but the framework becomes more involved and further assumptions are needed.

Section 3 is finally devoted to the non homogeneus problem (3); these results bring further the analysis done in the previous paper [3] and are preliminary to the study of the semilinear case (cfr. [4]).

1. THE COMMUTATIVE CASE

THEOREM 1. *Let the hypotheses* H_1) *be verified and assume that*

$$(4) \qquad [(n - C)^{-1}, (n - B)^{-1}] = 0 \qquad n > \sup \{\alpha_C , \alpha_B\}$$

then, if u_0 *is a* $\mathcal{F}_s$*-measurable random variable such that* $u_0 \in D_A$ a.e. *the process*

$$(5) \qquad u(t) = \exp \{C(t - s)\} \exp \{B(w_t - w_s)\} u_0$$

is the unique solution to the problem (1).

Proof. As $u_0 \in D_A$, by a simple application of the Itô formula, it easy to verify that (5) has a stochastic differential given by (1). Concerning the uniqueness let $u(t)$ be a solution of (1); then putting

$$v(t) = e^{-B(w_t - w_s)} u(t)$$

we have that $v(t)$ is differentiable (w.p. 1) and it is

$$v'(t) = Cv(t) \qquad v(s) = u_0$$

and, consequently, $v(t) = e^{C(t-s)} u_0$. Hence $u(t)$ is given by (5).

2. THE NON COMMUTATIVE CASE

Let u be a solution to problem (1); put:

$$(6) \qquad v(t) = \exp(- Bw_t) u(t)$$

then a formal use of the Itô formula gives:

$$(7) \qquad v'(t) = \exp(- Bw_t) C \exp(Bw_t) v(t)$$

so that we are led to study the following problem a.s.

$$(7) \qquad v'(t) = G(t) v(t) \qquad v(0) = u_0$$

where

$$(8) \qquad D_{G(t)} = \exp(- Bw_t) D_C \qquad G(t) x = \exp(- Bw_t) C \exp(Bw_t) x .$$

We remark that $\forall t \geq 0, w \in \Omega$ $G(t)$ is the infinitesimal generator of a C_0-semi-group on H:

$$\exp(G(t)s) = \exp(-Bw_t)\exp(Cs)\exp(Bw_t) \qquad s \geq 0 .$$

To treat problem (7) we give the following definition (see [6]):

DEFINITION 2. *The family* $G(t)$ *is* (M, α)*-stable on the subspace* $E \subset H$ *if*

$$\left|\prod_{i=1}^{n} \exp(-Bw_{t_i})\exp(Cs_i)\exp(Bw_{t_i})\right|_E \leq \tag{9}$$
$$\leq M \exp(\alpha(s_1 + \cdots + s_n)) \qquad \text{a.s.}$$

for all $t_1 \leq t_2 \leq \cdots \leq t_n, s_1, s_2, \cdots, s_n$.

We have the following result (see [2]):

PROPOSITION 3. *Let* (H_1) *be verified. Suppose that*:

(10) *the family* $G(t)$ *is* (M, α)*-stable in* H.

(11) *There exists a Hilbert space* $Y \hookrightarrow D_A \subset D_{G(t)}$ $\forall t \geq 0$ a.s. *such that the family* $G(t)/Y$ *is* (N, η)*-stable in* Y.

Then there exists the evolution operator $Z(t, s)$ *relative to* $G(t)$, *with the following properties*:

(12) $Z(t, s)$ *is strongly continuous*

(13) *if* $x \in Y$ *then* $t \to Z(t, s)x \in W^{1,p}([s, T]; H) \cap L^\infty(s, T; Y)$ (4)

(14) *let* $Z_n(t, s)$ *be the evolution operator relative to the family* $G_n(t) = \exp(-Bw_t)C_n\exp(Bw_t)$, $C_n = nC(n - C)^{-1}$, *then it is*

$$Z_n(t, s)x \xrightarrow{H} Z(t, s)x \qquad \forall x \in H$$

$$Z_n(t, s)x \xrightarrow{Y} Z(t, s)x \qquad \forall x \in Y .$$

Remark 4. The fact that $Y \hookrightarrow D_A$ yields

$$|G(t)|_{\mathscr{L}(Y,X)} \leq \beta$$

and it is just this property that it is used in [2].

Remark 5. By (14) we have that the function $t \to Z(t, s)x$ is non-anticipating.

(4) $W^{1,p}([s, T]; H) = \{u \in L^p(s, T; H)$ such that du/dt (in the distribution sense) $\in L^p(s, T; H)\}$.

Then the following theorem can be proved as Theorem 1.

THEOREM 6. *Let us suppose that the assumptions of Proposition* 1 *are fulfilled with* $Y \hookrightarrow D_A$. *Then if* $u_0 \in Y$ a.s. *problem* (1) *has a unique solution given by*:

$$u(t) = \exp(Bw_t)\, Z(t, s) \exp(-Bw_s)\, u_0 . \tag{15}$$

Remark 7. From (15) it follows (see [6])

$$|u(t)|_H \leq K \exp(\alpha_B |w_t| + \alpha_t) \tag{16}$$

thus if $\alpha < 0$, u is exponentially asymptotically stable a.s.

Remark 8. The stability condition is, in general, a delicate one. However a simple case in which it is verified is that of $\alpha_B = 0$, $M_B = M_C = 1$. In this case, in fact, $G(t)$ is $(1, \alpha_C)$–stable.

Exemple 9. Consider the following problem:

$$du(t) = (1/2 + \alpha(x))\, u_{xx}\, dt + u_x\, dw_t \qquad t \geq 0 , x \in R$$

where $\alpha \in C^2(R)$, $\alpha(x) \geq \alpha_0 > 0$ and $|\alpha''|$ bounded.

Put $H = L^2(R)$ and:

$$D_A = \{u \in H ; (1/2 + \alpha(x))\, u_{xx} \in H\} \quad , \quad (Au)(x) = (1/2 + \alpha(x))\, u_{xx}$$

$$D_B = \{u \in H ; u_x \in H\} \quad , \quad (Bu)(x) = u_x$$

$$D_C = \{u \in H ; \alpha(x)\, u_{xx} \in H\} \quad , \quad (Cu)(x) = \alpha(x)\, u_{xx}$$

then the hypotheses of Theorem 2 are verified with $Y = H^2(R)$. In fact it easy to check that $G(t)$ is $(1, \alpha_C)$–stable in both H and Y with $\alpha_C = 1/2 \sup |\alpha''|$.

3. THE INHOMOGENEOUS PROBLEM

We now consider problem (3) and look for a solution in the following sense:

DEFINITION 10. *We say that the process* $u(t)$ *is a strong solution to* (3) *if there exist sequences* u_n, φ_n, ψ_n *such that*:

$$u_n \to u , \varphi_n \to 0 , \psi_n \to 0 \qquad \text{in probability} \tag{17}$$

$$du_n = (Au_n + \varphi_n + f)\, dt + (Bu_n + g + \psi_n)\, dw_t \tag{18}$$

we have:

THEOREM 11. *Let the assumptions of Proposition* 3 *be verified. Assume* $f \in L^2_w(0, T; H)$, $g \in L^2_w(0, T; D_B)$ [5] *then problem* (3) *has a unique strong solution given by*:

$$u(t) = \exp(Bw_t)\Big[Z(t, 0)\, u_0 + \int_0^t Z(t, s) \exp(-Bw_s) \cdot \tag{19}$$

$$\cdot (f(s) - Bg(s))\, ds + \int_0^t Z(t, s) \exp(-Bw_s)\, g(s)\, dw_s\Big].$$

Proof. First of all we prove the theorem when $u_0 \in Y$ a.s. $f \in L^2_w(O, T; Y)$ $g \in L^2_w(O, T; Y)$, $Bg \in L^2_w(O, T; Y)$. Indeed it can easily checked that the solution of the following problem

$$du_n = A_n u_n\, dt + B_n u_n\, dw_t + f dt + g dw_t \tag{20}$$

is given by:

$$u_n = \exp(B_n w_t)\Big[Z_n(t, 0)\, u_0 + Z_n(t, 0) \int_0^t [Z_n(0, s) \exp(-B_n w_s) \cdot \tag{21}$$

$$\cdot (f(s) + Bg(s))\Big]\, ds + Z_n(t, 0) \int_0^t [Z_n(0, s) \exp(-B_n w_s)\, g(s)]\, dw_s\}$$

so that going to the limit in (20) and (21) it can be shown that (19) is the solution of (3). In the general case let $f_n \in Y$, $g_n \in Y$, $h_n \in Y$, $u_{0_n} \in Y$ such that

$$f_n \to f,\ g_n \to g,\ h_n \to Bg,\ u_{0_n} \to u_0 \qquad \text{is probability}$$

then

$$u_n = \exp(Bw_t)\Big[Z(t, 0)\, u_{0_n} + \int_0^t Z(t, s) \exp(-Bw_s)(f_n(s) - h_n)\, ds +$$

$$+ \int_0^t Z(t, s) \exp(-Bw_s)\, g_n(s)\, dw_s\Big]$$

verify (17) and (18) with

$$\varphi_n = -f + f_n - Bg - h_n \quad , \quad \psi_n = -g + g_n .$$

Remark 12. The above results say nothing about continuity properties of the solution of (3). We note, however, that if C is the infinitesimal generator of a C_0-group then continuity of $u(t)$ is evident from formula (19).

(5) Cfr. [5].

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