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On non-oscillatory behaviour of solutions of nonlinear differential equations

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Equazioni differenziali ordinarie. — *On non-oscillatory behaviour of solutions of nonlinear differential equations* (*). Nota (**) di N. PARHI e S. K. NAYAK, presentata dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota si danno condizioni sufficienti perchè tutte le soluzioni delle equazioni

$$(r(t)\,y')' - p(t)\,y^{\nu} = f(t) \quad \text{ed} \quad y'' - g(t,y) = f(t)$$

siano non oscillatorie.

1. INTRODUCTION

In this paper, we are concerned with the nonoscillatory behaviour of solutions of the following second order nonlinear differential equations;

$$(r(t)\,y')' - p(t)\,y^{\nu} = f(t) \tag{1.1}$$

$$y'' - g(t,y) = f(t), \tag{1.2}$$

where r, p and f are real-valued continuous functions on $[0, \infty)$ such that $r(t) > 0$, $p(t) \geq 0$, $f(t) \geq 0$ for $t \in [0, \infty)$ and g is a real valued continuous function defined for $t \in [0, \infty)$, $y \in (-\infty, \infty)$ and $\nu > 0$ is the ratio of odd integers. We restrict our consideration to those solutions of (1.1), (1.2) which exist on the half-line $[T, \infty)$, where T may depend on the particular solution, and are non-trivial in any neighbourhood of infinity. In [2], we studied the nonoscillatory behaviour of solutions of $y'' - p(t)\,y = f(t)$ and $y'' - p(t)\,y^{\nu} = f(t)$.

We classify solutions of (1.1) or (1.2) as follows: a solution $y(t)$ is said to be nonoscillatory if there exists a $t_1 \geq T$ such that $y(t) \neq 0$ for $t \geq t_1$; $y(t)$ is said to be oscillatory if for any $t_1 \geq T$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$ such that $y(t_2) > 0$ and $y(t_3) < 0$; and it is said to be a Z-type solution if it has arbitrarily large zeros but is ultimately non-negative or non-positive.

2. In this section, we obtain sufficient conditions so that all solutions of Eqs. (1.1) or (1.2) are non-oscillatory.

THEOREM 2.1. *Consider* Eq. (1.1). *Let rp and rf be once continuously differentiable. Let $(rp)'(t) \leq 0$, $(rf)'(t) \geq 0$ and rp and rf be not constants over a common interval. Then all solutions of* (1.1) *are nonoscillatory.*

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Proof. It follows from concavity considerations that no solution of (1.1) is oscillatory or non-negative Z-type. Let $y(t)$ be a non-positive Z-type solution of (1.1) with consecutive double zeros at a and b. Multiplying (1.1) through by $r(t)\,y'(t)$ we get

$$\frac{d}{dt}\left[\frac{1}{2}(r(t)\,y'(t))^2-\frac{1}{\nu+1}\,r(t)\,p(t)\,y^{\nu+1}(t)-r(t)\,f(t)\,y(t)\right]$$
$$=-\frac{1}{\nu+1}\,(r(t)\,p(t))'\,y^{\nu+1}(t)-(r(t)\,f(t))'\,y(t)\,.$$

Integrating above equality from a to b we get

$$0=-\frac{1}{\nu+1}\int_a^b(r(t)\,p(t))'\,y^{\nu+1}(t)\,dt-\int_a^b(r(t)\,f(t))'\,y(t)\,dt>0$$

a contradiction. Hence the theorem.

THEOREM 2.2. *Consider Eq.* (1.1). *Let r, p and f be once continuously differentiable such that $r'(t)\leq 0$, $p'(t)\leq 0$ and $f'(t)\geq 0$. Let $p(t)$ and $f(t)$ be not constants over a common interval. Then all solutions of* (1.1) *are nonoscillatory.*

Proof. It is clear from concavity considerations that no solution of (1.1) is oscillatory or non-negative Z-type. Let $y(t)$ be a non-positive Z-type solution of (1.1) with consecutive double zeros at a and b. Multiplying (1.1) through by $y'(t)$ we get

$$\frac{d}{dt}\left[\frac{1}{2}r(t)\,(y'(t))^2-\frac{1}{\nu+1}\,p(t)\,y^{\nu+1}(t)-f(t)\,y(t)\right]$$
$$=-\frac{1}{\nu+1}\,p'(t)\,y^{\nu+1}(t)-f'(t)\,y(t)-\frac{1}{2}r'(t)\,(y'(t))^2.$$

Integrating the above equality from a to b we get

$$0=-\frac{1}{\nu+1}\int_a^b p'(t)\,y^{\nu+1}(t)\,dt-\int_a^b f'(t)\,y(t)\,dt$$
$$-\frac{1}{2}\int_a^b r'(t)\,(y'(t))^2\,dt>0\,,$$

a contradiction. This completes the theorem.

Remark 2.3. Theorem 2.2 is true if $r'(t)\geq 0$, $p'(t)\geq 0$ and $f'(t)\leq 0$.

In the following we prove a theorem for $\nu\geq 1$ in which we show that all solutions of (1.1) are nonoscillatory with only a continuity condition on r, p and f.

THEOREM 2.4. *Consider Eq.* (1.1) *with* $\nu \geq 1$. *All solutions of* (1.1) *are nonscillatory if* $f(t) > 0$.

Proof. To complete the proof of the theorem it is enough to show that no solution of (1.1) can be of non-positive Z-type because from concavity considerations it follows that no solution of (1.1) is oscillatory or of non-negative Z-type. If possible let $y(t)$ be a non-positive Z-type solution of (1.1) with consecutive double zeros at a and b $(a < b)$. So there exists a $c \in (a, b)$ such that $y'(t) < 0$ for $t \in (a, c]$. Let $\varepsilon > 0$ be such that $a + \varepsilon < c$. For $t \in [a + \varepsilon, c]$, Eq. (1.1) can be written as

$$(r(t)\, y'(t))'\, y^{-\nu}(t) = p(t) + f(t)\, y^{-\nu}(t)\,. \tag{2.1}$$

Integrating (2.1) from $a + \varepsilon$ to c we get

$$[r(t)\, y'(t)\, y^{-\nu}(t)]_{a+\varepsilon}^{c} + \nu \int_{a+\varepsilon}^{c} y^{-\nu-1}(t)\, r(t)\, (y'(t))^2\, \mathrm{d}t = \int_{a+\varepsilon}^{c} p(t)\, \mathrm{d}t + \int_{a+\varepsilon}^{c} f(t)\, y^{-\nu}(t)\, \mathrm{d}t\,,$$

that is,

$$-r(a+\varepsilon)\, y'(a+\varepsilon)\, y^{-\nu}(a+\varepsilon) \leq \int_{a+\varepsilon}^{c} p(t)\, \mathrm{d}t\,. \tag{2.2}$$

But

$$\lim_{\varepsilon \to 0} \frac{r(a+\varepsilon)\, y'(a+\varepsilon)}{y^{\nu}(a+\varepsilon)} = \lim_{\varepsilon \to 0} \frac{p(a+\varepsilon)\, y^{\nu}(a+\varepsilon) + f(a+\varepsilon)}{\nu y^{\nu-1}(a+\varepsilon)\, y'(a+\varepsilon)} = -\infty\,.$$

Hence taking the limit, as $\varepsilon \to 0$, in (2.2) we get $\int_{a}^{c} p(t)\, \mathrm{d}t \geq \infty$, a contradiction. This proves the theorem.

Remark 2.5. A solution $y(t)$ of Eq. (1.1), under the conditions of Theorem 2.4, can have at most two zeros of the type $y(a) = 0 = y(b)$, $y'(a) < 0$, $y'(b) > 0$.

In the following, we prove a theorem which may be viewed as a complement to Theorem 2.4.

THEOREM 2.6. *Consider Eq.* (1.1) *with* $p(t) > 0$. *If* $f(t) \geq p(t)$, *then all solutions of* (1.1) *are nonoscillatory.*

Proof. We complete the proof of the theorem by showing that no solution of (1.1) is of non-positive Z-type. If possible, let $y(t)$ be a non-positive Z-type solution of (1.1) with consecutive double zeros at a and b $(a < b)$. So there exists a point $c \in (a, b)$ such that $y(t) > -1$ for $t \in [c, b]$. Let $t \in [c, b]$.

Integrating (1.1) from t to b we get

$$-r(t)\,y'(t) = \int_t^b (f(s) + p(s)\,y^{\nu}(s))\,ds\,.$$

Integrating the above equality from t to b we get

$$y(t) = \int_t^b \frac{ds}{r(s)} \int_s^b (f(u) + p(u)\,y^{\nu}(u))\,du$$

$$= \int_t^b \left(\int_t^s \frac{du}{r(u)}\right)(f(s) + p(s)\,y^{\nu}(s))\,ds\,.$$

Since $y(t) \leq 0$, we have

$$\int_t^b \left(\int_t^s \frac{du}{r(u)}\right)(f(s) + p(s)\,y^{\nu}(s))\,ds \leq 0\,.$$

So there exists a point $t_1 \in (t\,,b)$ such that $f(t_1) + p(t_1)\,y^{\nu}(t_1) \leq 0$, that is, $f(t_1) < p(t_1)$, since $y^{\nu}(t_1) > -1$. This contradiction proves the theorem.

Remark 2.7. A solution $y(t)$ of (1.1), under the conditions of Theorem 2.6, can have at most two zeros of the type $y(a) = 0 = y(b)$, $y'(a) < 0$, $y'(b) > 0$.

The following examples justify Theorems 2.2 and 2.6.

Example 2.8. Consider

$$y'' - y^{1/3} = 1/t - 12/t^5\,, \qquad t \geq 3\,. \tag{2.3}$$

From Theorem 2.2 and Remark 2.3, it is clear that all solutions of (2.3) are nonoscillatory. In particular, $y(t) = -1/t^3$ is a nonoscillatory solution of (2.3). It is easy to note that Theorem 2.6 cannot be applied to Eq. (2.3) since $1 > 1/t - 12/t^5$ for $t \geq 3$.

Example 2.9. Consider

$$y'' - y^{1/3} = 1\,, \qquad t \geq 0\,. \tag{2.4}$$

All solutions of (2.4) are nonoscillatory by Theorem 2.6. $y(t) = -1$ is a particular nonoscillatory solution of (2.4). Theorem 2.2 cannot be applied to (2.4) since $p(t) = f(t) = 1$.

Lastly, we prove a theorem similar to the above theorems for Eq. (1.2).

THEOREM 2.10. *Consider Eq.* (1.2). *Let f be once continuously differentiable such that $f'(t) \geq 0$. Let $yg(t\,,y) > 0$ if $y \neq 0$. Further, assume that* *there exists a function* $h(t, y)$, $t \geq 0$ *and* $y \in (-\infty, \infty)$, *such that* $h(t, 0) = 0$, $h_t(t, y) \leq 0$ *and* $h_y(t, y) = g(t, y)$. *Then all solutions of* (1.2) *are nonoscillatory.*

Proof. Clearly, $yg(t, y) > 0$, $y \neq 0$ implies that no solution of (1.2) is oscillatory or non-negative Z-type. Let $y(t)$ be a non-positive Z-type solution of (1.2) with consecutive double zeros at a and $b\,(a < b)$. Multiplying (1.2) through by $y'(t)$ we get

$$\frac{d}{dt}\left[\tfrac{1}{2}(y'(t))^2 - h(t, y(t)) - f(t)\,y(t)\right] = -h_t(t, y(t)) - f'(t)\,y(t).$$

Integrating above equality from a to b we get

$$0 = -\int_a^b h_t(t, y(t))\,dt - \int_a^b f'(t)\,y(t)\,dt > 0,$$

a contradiction. Hence the theorem.

Remark 2.11. A solution $y(t)$ of Eq. (1.2) under the hypothesis of Theorem 2.10 may have at most two zeros of the type $y(a) = 0 = y(b)$, $y'(a) \leq 0$, $y'(b) > 0$, $a < b$.

References

[1] E. Hille (1970) - *Lectures on Ordinary Differential Equations*, Addison-Wesley, London, 1970.

[2] N. Parhi and S. K. Nayak - *On the Behaviour of Solutions of* $y'' - p(t)\,y^{\nu} = f(t)$, to apper in «Applicable Analysis».