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A further result on the existence of periodic solutions of the equation $\bar{x} + \psi(\dot{x})\ddot{x} + \phi(x)\dot{x} + \theta(t, x, \dot{x}, \ddot{x}) = p(t)$ with a bounded θ

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1978.

Equazioni differenziali ordinarie. — *A further result on the existence of periodic solutions of the equation* $\dddot{x} + \psi(\dot{x})\ddot{x} + \phi(x)\dot{x} + \theta(t, x, \dot{x}, \ddot{x}) = p(t)$ *with a bounded* θ. Nota (*) di JAMES O.C. EZEILO, presentata dal Socio G. SANSONE.

RIASSUNTO. — L'Autore generalizza alcuni risultati ottenuti in precedenza da R. Reissig.

1. In the equation in the title which will hereafter be referred to as the equation (R), the functions ψ, ϕ, θ and p are continuous functions depending only on the arguments shown and θ, p are ω-periodic in t, that is

$$\theta(t+\omega, x, y, z) = \theta(t, x, y, z)$$

and $p(t+\omega) = p(t)$ for arbitrary x, y, z, t and for some $\omega > 0$. Let $\Phi(x) \equiv \equiv \int_0^x \phi(\xi)\,d\xi$ and $\Psi(y) \equiv \int_0^y \psi(\eta)\,d\eta$. The case in which θ is bounded in (R), that is

$$(1.1) \qquad |\theta(t, x, y, z)| \leq F \text{ (constant)} \quad \text{for all } x, y, z \text{ and } t,$$

has been the object of a special study by Reissig [1] who showed that if

$$(1.2) \qquad \theta(t, x, y, z)\,\mathrm{sgn}\,x \geq 0 \quad \text{for } |x| \geq h > 0$$

and if further the following conditions hold:

$$(1.3) \qquad \int_0^\omega p(t)\,dt = 0$$

$$(1.4) \qquad \Psi(y)\,\mathrm{sgn}\,y \to \infty \quad \text{as } |y| \to \infty$$

$$(1.5) \qquad |\Phi(x) - bx| \leq M \text{ (constant)} \quad \text{for all } x,$$

for some constant $b > 0$, then the equation (R) has at least one ω-periodic solution.

The object of the present note is to show that this existence result of Reissig does hold under a much weaker condition on ψ than (1.4) and, what is even more important, without the use of (1.5) or indeed any other restriction on ϕ.

(*) Pervenuta all'Accademia il 4 agosto 1978.

Let $A \equiv \max_{0 \leq t \leq \omega} |p(t)|$. We shall in fact prove that:

THEOREM. *If* (1.1) (1.2) *and* (1.3) *hold and if further there are constants* $\eta_0 > 0$, $\delta_0 > (F + A)\omega$ *such that*

$$\Psi'(y) \operatorname{sgn} y \geq \delta_0 \quad \textit{for} \quad |y| \geq \eta_0, \tag{1.6}$$

then the equation (R) *has at least one* ω*-periodic solution for all arbitrary* ϕ.

Note that, by (1.6)

$$y\Psi'(y) - \delta_0 |y| \geq 0 \qquad \text{for} \quad |y| \geq \eta_0$$

and that, if $\delta_1 \equiv \delta_0 \eta_0 + \eta_0 \max_{|y| \leq \eta_0} |\Psi'(y)|$,

$$y\Psi'(y) + \delta_1 - \delta_0 |y| \geq 0 \qquad \text{for} \quad |y| \leq \eta_0$$

so that, at least,

$$y\Psi'(y) \geq \delta_0 |y| - \delta_1 \qquad \text{for all} \quad y. \tag{1.7}$$

We shall find it convenient in our proof to use (1.7) directly rather than (1.6).

2. The procedure here will be on the lines of the Leray-Schauder principle in the form suggested by Güssefeldt [2] and as outlined in [1] except that in place of Reissig's parameter dependent equation ((6) of [1]) we shall consider the equation

$$\dddot{x} + \psi_\mu(\dot{x})\ddot{x} + \phi_\mu(x)\dot{x} + g(t, x, \dot{x}, \ddot{x}) = \mu p(t) \tag{2.1}$$

where, as before, μ is in the half open range

$$0 \leq \mu < 1 \tag{2.2}$$

and g is defined by

$$g(t, x, \dot{x}, \ddot{x}) = \mu\theta(t, x, \dot{x}, \ddot{x}) + (1 - \mu)\frac{Fx}{1 + |x|}$$

(so that $|g| \leq F$ always), but ψ_μ and ϕ_μ are now given by

$$\psi_\mu(\dot{x}) = \mu\psi(\dot{x}) + (1 - \mu)\delta_0 \tag{2.3}$$

$$\phi_\mu(x) = \mu\phi(x). \tag{2.4}$$

It should be pointed out here that our ψ_μ, which is different from that used in [1], has been specially chosen with the new restriction (1.6) on ψ in view. As for the function ϕ_μ, Reissig's $\phi(x, \mu)$ could have done just as well in its place and the only advantage in our (2.4) is its relative simplicity.

The crucial point to note, however, is that, at $\mu = 0$, (2.1) reduces to the equation:

$$\dddot{x} + \delta_0 \ddot{x} + \frac{Fx}{1 + |x|} = 0$$

an important property of which is that it may be rewritten in the "split" form:

$$\dddot{x} + \delta_0 \ddot{x} + cx + \left\{ \frac{Fx}{1 + |x|} - cx \right\} = 0 \tag{2.5}$$

with the term inside the curly brackets in (2.5) an odd function of x if c is a constant, while the equation

$$\dddot{x} + \delta_0 \ddot{x} + cx = 0 ,$$

arising from the remaining terms in (2.5) has no non-trivial ω-periodic solution if $c \neq 0$ since the corresponding auxilliary equation:

$$\lambda^3 + \delta_0 \lambda^2 + c = 0$$

has no purely imaginary roots. Thus all of Reissig's exposition in [1] of the application of Güssefeldt's form of the Leray-Schauder principle holds good here, and it will indeed be enough for our proof of the theorem to show that, subject to (1.1), (1.2), (1.3) and (1.6) alone, there are positive constants D_0, D_1, D_2 *all independent of* μ such that every ω-periodic solution $x(t)$ of (2.1), with μ subject to (3.2), satisfies

$$|x(t)| \leq D_0 , \ |\dot{x}(t)| \leq D_1 , \ |\ddot{x}(t)| \leq D_2 \qquad (0 \leq t \leq \omega) . \tag{2.6}$$

3. Throughout what follows the capitals D, with or without suffixes, denote positive constants, whose magnitudes depend on δ_0, δ_1, h, F, A, ψ and ϕ but not on μ. The D's without suffixes are not necessarily the same in each place of occurrence, but the D's: D_3, D_4, $\cdots$ with suffixes attached retain a fixed identity throughout.

Note that if $\Psi_\mu(y) \equiv \int_0^y \psi_\mu(\eta)\, d\eta$ where ψ_μ is defined by (2.3) so that indeed

$$y\Psi_\mu(y) = (1 - \mu)\, \delta_0\, y^2 + \mu y \Psi(y)$$

then, since $y^2 \geq |y| - 1/4$ for all y, we shall have from (1.7) and analagous to it, that

$$y\Psi_\mu(y) \geq \delta_0 |y| - D_3 \qquad \text{for all } y \tag{3.1}$$

and for some D_3.

The following two results which hold for any ω-periodic solution $x(t)$ of (2.1) subject to (2.2):

$$\max_{0 \le t \le \omega} |x(t)| \le h + \int_{\tau_0}^{\tau_0+\omega} |\dot{x}(t)| \, dt \,, \tag{3.2}$$

$$\int_{\tau_1}^{\tau_1+\omega} |x(t)| \, dt \le \omega \left(h + \int_{\tau_1}^{\tau_1+\omega} |\dot{x}(t)| \, dt \right) . \tag{3.3}$$

will also be vital to our proofs. The τ_0, τ_1 here may be taken arbitrarily since

$$\int_{\tau_0}^{\tau_0+\omega} |\dot{x}(t)| \, dt = \int_0^{\omega} |\dot{x}(t)| \, dt = \int_{\tau_1}^{\tau_1+\omega} |\dot{x}(t)| \, dt$$

(with a corresponding result for $\int |x(t)| \, dt$) if $x(t)$ is ω-periodic. To verify (3.2) we first observe, on integrating (2.1) fron $t = 0$ to $t = \omega$ and using (1.3) that

$$\int_0^{\omega} g(t, x, \dot{x}, \ddot{x}) \, dt = 0 . \tag{3.4}$$

Next, it is clear from the definition of g that $g(t, x, \dot{x}, \ddot{x}) \operatorname{sgn} x > 0 \; (|x| \ge h)$ if $(1 - \mu) > 0$ so that (3.4) necessarily implies that $|x(\tau_0)| \le h$ for some $\tau_0 \in [0, \omega]$ from which (3.2) then follows on applying the identity: $x(t) = x(\tau) + \int_{\tau}^{t} \dot{x}(s) \, ds$. The inequality (3.3) is a direct consequence of (3.2).

We turn now to the actual verification of (2.6) and start with the estimate for $|x(t)|$, $x = x(t)$ being an arbitrary ω-periodic solution of (2.1) with μ subject to (2.2) Let us multiply both sides of (2.1) by x and integrate with respect to t. Since

$$\int x\dddot{x} \, dt = x\ddot{x} - \tfrac{1}{2}\dot{x}^2 \,, \int x\psi_{\mu}(\dot{x}) \, \ddot{x} dt = x\Psi_{\mu}(\dot{x}) - \int \dot{x}\Psi_{\mu}(\dot{x}) \, dt$$

and

$$\frac{d}{dt} \int_0^x \xi\phi_{\mu}(\xi) \, d\xi = \phi_{\mu}(x) \, x\dot{x}$$

the integration gives, x being ω-periodic, simply that

$$\int_0^{\omega} \dot{x}\Psi_{\mu}(\dot{x}) \, dt = \int_0^{\omega} xg dt - \mu \int_0^{\omega} xp dt .$$

Hence, by (3.1) and since $|g| \leq F$ and $|p| \leq A$, we have that

$$\delta_0 \int_0^\omega |\dot{x}|\,dt \leq D + (F + A) \int_0^\omega |x|\,dt,$$

$$\leq D_4 + (F + A)\,\omega \int_0^\omega |\dot{x}|\,dt,$$

by (3.3). Thus if $\delta_0 > (F + A)\,\omega$ as given in the theorem, then clearly

$$\int_0^\omega |\dot{x}|\,dt \leq D_4 \{\delta_0 - (F + A)\,\omega\}^{-1} \equiv D_5, \tag{3.5}$$

say. Hence, by (3.2),

$$\max_{0 \leq t \leq \omega} |x(t)| \leq h + D_5 \tag{3.6}$$

which establishes the first estimate in (2.6).

To obtain the second of the estimates in (2.6), multiply (2.1) next by $\dot{x}$ and integrate with respect to t. Since

$$\int \dot{x}\dddot{x}\,dt = \dot{x}\ddot{x} - \int \ddot{x}^2\,dt\,,\ \int \phi(x)\,\dot{x}^2\,dt = \dot{x}\Phi(x) - \int \ddot{x}\Phi(x)\,dt$$

and

$$\frac{d}{dt} \int_0^{\dot{x}} \psi_\mu(s)\,s\,ds = \psi_\mu(\dot{x})\,\dot{x}\ddot{x}$$

we obtain as a result of the integration, and using the ω-periodicity of x, that

$$\int_0^\omega \ddot{x}^2\,dt = -\mu \int_0^\omega \ddot{x}\Phi(x)\,dt - \int_0^\omega (\mu p - g)\,\dot{x}\,dt. \tag{3.7}$$

Recall that $0 \leq \mu < 1$, so that in particular

$$\left| \int_0^\omega (\mu p - g)\,\dot{x}\,dt \right| \leq (F + A) \int_0^\omega |\dot{x}|\,dt$$

$$\leq D,$$

by (3.5). Also by (3.6), $|\Phi(x)| \leq D$, so that (3.7) implies that

$$\int_0^\omega \ddot{x}^2\,dt \leq D \left(\int_0^\omega |\ddot{x}|\,dt + 1 \right) \tag{3.8}$$

$$\leq D \left\{ \left(\int_0^\omega \ddot{x}^2\,dt \right)^{1/2} + 1 \right\},$$

by Schwarz's inequality. It is clear from (3.8) that

$$\int_0^\omega \dddot{x}^2 \,\mathrm{d}t \leq \mathrm{D}\,. \tag{3.9}$$

Then, since $\dot{x}(\tau_0) = 0$ necessarily at some $\tau_0 \in [0, \omega]$ (in view of the ω–periodicity condition: $x(0) = x(\omega)$) it follows from the identity:

$$\dot{x}(t) = \dot{x}(\tau) + \int_\tau^t \ddot{x}(s)\,\mathrm{d}s$$

with $\tau = \tau_0$ that

$$\dot{x}(t) = \int_{\tau_0}^t \ddot{x}(s)\,\mathrm{d}s\,,$$

so that

$$\max_{0 \leq t \leq \omega} |\dot{x}(t)| \leq \int_{\tau_0}^{\tau_0+\omega} |\ddot{x}(s)|\,\mathrm{d}s \tag{3.10}$$
$$\leq \omega^{1/2}\left(\int_{\tau_0}^{\tau_0+\omega} \ddot{x}^2(s)\,\mathrm{d}s\right)^{1/2}$$
$$\leq \mathrm{D}\,,$$

by (3.9), which establishes the second estimate in (2.6).

For the remaining estimate in (2.6) we will multiply (2.1) this time by $\dddot{x}$ but integrate again with respect to t from $t = 0$ to $t = \omega$. Once it is recalled that $|g| \leq \mathrm{F}$, $|p| \leq \mathrm{A}$ and then that

$$|\psi_\mu(\ddot{x})| \leq \mathrm{D} \qquad \text{and} \qquad |\mu\phi(x)\dot{x}| \leq \mathrm{D}$$

as a result of (2.2), (3.6) and (3.10), it is easy to see from this last integration that

$$\int_0^\omega \dddot{x}^2\,\mathrm{d}t \leq \mathrm{D}\left(\int_0^\omega |\ddot{x}|\,|\dddot{x}|\,\mathrm{d}t + \int_0^\omega |\dddot{x}|\,\mathrm{d}t\right),$$

so that, by Schwarz's inequality,

$$\int_0^\omega \dddot{x}^2\,\mathrm{d}t \leq \mathrm{D}\left(\int_0^\omega \dddot{x}^2\,\mathrm{d}t\right)^{1/2}\left\{1 + \left(\int_0^\omega \ddot{x}^2\,\mathrm{d}t\right)^{1/2}\right\} \tag{3.11}$$
$$\leq \mathrm{D}\left(\int_0^\omega \dddot{x}^2\,\mathrm{d}t\right)^{1/2}$$

because of (3.9). It is clear from (3.11) that

$$(3.12) \qquad \int_0^{\omega} \dddot{x}^2 \, dt \leq D \,.$$

The usual consideration of the fact that $\ddot{x}(\tau_1) = 0$ necessarily for some $\tau_1 \in [0, \omega]$ (in view of the ω-periodicity condition $\dot{x}(0) = \dot{x}(\omega)$) side by side with the identity

$$\ddot{x}(t) = \ddot{x}(\tau_1) + \int_{\tau_1}^{t} \dddot{x}(s) \, ds$$

then gives that

$$\max_{0 \leq t \leq \omega} |\ddot{x}(t)| \leq \int_{\tau_1}^{\tau_1+\omega} |\dddot{x}(s)| \, ds$$

$$\leq \omega^{1/2} \left(\int_{\tau_1}^{\tau_1+\omega} \dddot{x}^2(s) \, ds \right)^{1/2}$$

$$\leq D$$

by (3.12).

This completes the verification of (2.6) for any arbitrary ω-periodic solution $x(t)$ of (2.1) in which the parameter μ is subject to (2.2). The theorem then follows as pointed out in § 2.

4. By way of conclusion it may be noted that Reissig's observation that his existence theorem remains true if we assume that $\theta(t, x, y, z) \operatorname{sgn} x \leq 0$ (instead of ≥ 0) for $|x| \geq h$ equally applies here, and for precisely the same reasons as were given in [1]. Also, analagous to Reissig's Remark 2 (on page 203 of [1]) and for the same reasons as in [1], the existence theorem here remains true if we assume that

$$\Psi(y) \operatorname{sgn} y \leq -\delta_0 \text{ (instead of } \geq \delta_0) \text{ for } |y \geq \eta_0 \text{ where } \delta_0 > (F + A)\,\omega \,.$$

REFERENCES

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