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Asymptotic behaviour of perturbed difference equations

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Equazioni funzionali. — *Asymptotic behaviour of perturbed difference equations.* Nota II di PAVEL TALPALARU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore con l'impiego di una disuguaglianza (Lemma 3.1) dà alcuni risultati concernenti l'equivalenza asintotica di equazioni alle differenze.

1. INTRODUCTION

Difference equations occur in many branches of applied mathematics: numerical analysis, physics, sampled-data systems, control theory and optimization.

In order to discuss the asymptotic problems for difference equations, generally speaking, the analogous methods to those for differential and functional-differential equations are applicable and the results obtained are also analogous to those for differential and functional-differential equations.

The aim of the present paper is to extend part of the results of [3] concerning the asymptotic equivalence of integro-differential systems to the case of certain difference systems.

Fixed point technique and difference inequalities [1] have been used to prove our main results.

2. NOTATIONS AND DEFINITIONS

Denote by $N(n_0) = \{n_0, n_0 + 1, \cdots\}$, where n_0 is a natural number or zero; $R^{\bar{k}}$ the $\bar{k}$-dimensional real euclidean space with norm $|x| = \sum_{i=1}^{\bar{k}} |x_i|$, $x = (x_1, x_2, \cdots, x_{\bar{k}})$; $M^{\bar{k}}$ the space of all $\bar{k} \times \bar{k}$ matrices $A = (a_{ij})$ with norm $|A| = \max_j \sum_{i=1}^{\bar{k}} |a_{ij}|$. We denote by $\Phi = \Phi(N, R^{\bar{k}})$ the space of all functions from $N(n_0)$ into $R^{\bar{k}}$, that is, for each $n \in N(n_0)$ the value of x at n is $x(n) \in R^{\bar{k}}$. The topology of Φ is the topology of uniform convergence on every set $N_m(n_0) = \{n_0, n_0 + 1, \cdots, n_0 + m\}$, $m = 0, 1, \cdots$, that is, $x_i \to x$ as $i \to \infty$ in Φ if and only if $\lim_{i \to \infty} |x_i(n) - x(n)| = 0$ uniformly on every set $N_m(n_0)$, $m = 0, 1, \cdots$. Note also that Φ is locally convex space [5, pp. 24–26] with the topology defined by the following family of seminorms $|x(n)|_m = \sup\{|x(n)|; n \in N_m(n_0), m = 0, 1, \cdots\}$.

(*) Nella seduta del 13 maggio 1978.

Let $\Phi_1 = \Phi_1(N, R^{\bar{k}})$ be the Banach space in Φ of all bounded function from $N(n_0)$ to $R^{\bar{k}}$. The norm in Φ_1 is defined by $|x|_{\Phi_1} = |x(n)|_{\Phi_1} = \sup\{|x(n)|; n \in N(n_0)\}$.

The space Φ and Φ_1 was considered by D. Petrovanu [2] and C. P. Tsokos and W. J. Padgett [4, Ch. V] in the study of functional equations.

We will consider two systems

$$x(n+1) = A(n)\, x(n) \tag{2.1}$$

and

$$y(n+1) = A(n)\, y(n) + f(n, y(n)) + \sum_{s=n_0}^{n} k(n, s, y(s)), \tag{2.2}$$

where x, y are $\bar{k}$-dimensional vectors, $A: N(n_0) \to M^{\bar{k}}$ is such that $A(n)$ is nonsingular for all $n \in N(n_0)$, $f: N(n_0) \times D \to R^{\bar{k}}$ is, for any $n \in N(n_0)$ continuous as a function of $y \in D$ (D – a region in $R^{\bar{k}}$), $k: N(n_0) \times N(n_0) \times D \to R^{\bar{k}}$ is, for any $(n, s) \in N(n_0) \times N(n_0)$ continuous with respect to $y \in D$. Note that if $X(n)$ is the fundamental matrix of (2.1), then $X(n)$ is a nonsingular matrix and therefore $X^{-1}(n)$, exists for any $n \in N(n_0)$.

Observe that (2.2) is discrete analogous of a perturbed integro-differential equation. In the following we will be concerned with the study of the asymptotic equivalence of the systems (2.1) and (2.2). The notion of asymptotic equivalence which will be used is given by

DEFINITION 2.1. Let $\Delta(n)$ be $\bar{k} \times \bar{k}$ matrix, and $\alpha(n)$ a positive function; we say that the systems (2.1) and (2.2) are asymptotically equivalent if, corresponding to each solution $x = x(n)$ of system (2.1), there exists a solution $y = y(n)$ of (2.2) with the property that

$$|\Delta(n)[x(n) - y(n)]| = o(\alpha(n)) \quad , \quad n \to \infty, \tag{2.3}$$

and conversely, to each solution $y = y(n)$, of (2.2) there corresponds a solution $x = x(n)$ of (2.1) such that (2.3) holds.

3. A PRELIMINARY RESULT

To establish the main results on asymptotic equivalence we need to give the following lemma.

LEMMA 3.1. ([1]). *Let there exist functions $u(n)$, $v(n)$, $h(n, s)$ and $\omega(r)$ such that:*

a) $u(n)$, $v(n)$, $h(n, s)$ are non-negative for $n \geq s$, $n, s \in N(n_0)$;

b) $\omega(r)$ is positive, continuous and non-decreassing for $r > 0$;

c) for any $n \in N(n_0)$ we have the inequality

$$u(n) \leq c + \sum_{s=n_0}^{n-1} \left[v(s)\, \omega(u(s)) + \sum_{1=n_0}^{s} h(s, 1)\, \omega(u(1)) \right], \tag{3.1}$$

where c is a positive constant. Then for any $n \in \mathrm{N}(n_0)$ *we have*

$$\Omega(u(n)) \leq \Omega(c) + \sum_{s=n_0}^{n-1} \left[v(s) + \sum_{1=n_0}^{s} h(s, 1) \right], \tag{3.2}$$

where

$$\Omega(z) = \int^{z} \frac{dz}{\omega(z)}.$$

Proof. Define

$$b(n_0) = c, \; b(n) = c + \sum_{s=n_0}^{n-1} \left[v(s)\,\omega(u(s)) + \sum_{1=n_0}^{s} h(s, 1)\,\omega(u(1)) \right] \; n \in \mathrm{N}(n_0 + 1).$$

From *a*), *b*) and (3.1) it follows that $b(n) > 0$, $u(n) \leq b(n)$ and $\omega(u(n)) \leq \leq \omega(b(n))$ for $n \in \mathrm{N}(n_0)$. Then, since $b(1) \leq b(n)$ for $1 \leq n$ we have $\omega((u(1)) \leq \omega(u(n))$ and therefore

$$b(n+1) = v(n)\,\omega(u(n)) + \sum_{1=n_0}^{n} h(n, 1)\,\omega(u(1)) + b(n) \leq$$
$$\leq v(n)\,\omega(u(n)) + \omega(u(n)) \sum_{1=n_0}^{n} h(n, 1) + b(n),$$

from where

$$\frac{\Delta b(n)}{\omega(b(n))} = \frac{b(n+1) - b(n)}{\omega(b(n))} \leq v(n) + \sum_{1=n_0}^{n} h(n, 1), \; n \in \mathrm{N}(n_0). \tag{3.3}$$

If $b(s) \leq z \leq b(s+1)$, then $\omega(b(s)) \leq \omega(z) \leq \omega(b(s+1))$ and $[\omega(z)]^{-1} \leq [\omega(b(s))]^{-1}$, and this implies

$$\int_{b(s)}^{b(s+1)} \frac{dz}{\omega(b(s))} = \frac{\Delta b(s)}{\omega(b(s))} \geq \int_{b(s)}^{b(s+1)} \frac{dz}{\omega(z)}.$$

From here and (3.3) we get

$$\int_{b(s)}^{b(s+1)} \frac{dz}{\omega(z)} \leq v(s) + \sum_{1=n_0}^{s} h(s, 1), \; s \in \mathrm{N}(n_0),$$

hence,

$$\sum_{s=n_0}^{n-1} \int_{b(s)}^{b(s+1)} \frac{dz}{\omega(z)} = \int_{c}^{b(n)} \frac{dz}{\omega(z)} \leq \sum_{s=n_0}^{n-1} \left[v(s) + \sum_{1=n_0}^{s} h(s, 1)) \right].$$

The inequality (3.2) follows now using the definition of $\Omega(z)$ and $u(n) \leq b(n)$.

Remarks 3.1. When $h(s, 1) = 0$, the inequality (3.2) reduces to a discrete-time version of a certain form of the Bellman-Gronwall inequality.

3.2. If $\omega(r) = r$, the inequality (3.2) becomes

$$u(n) \leq c \exp\left\{\sum_{s=n_0}^{n-1}\left[v(s) + \sum_{1=n_0}^{s} h(s, 1)\right]\right\}.$$

3.3. Using *b*) it follows that there exists $z = \Omega^{-1}(w)$ and (3.8) may be written also as

$$(3.8') \qquad u(n) \leq \Omega^{-1}\left(\Omega(c) + \sum_{s=n_0}^{n-1}\left[v(s) + \sum_{1=n_0}^{s} h(s, 1)\right]\right).$$

4. Asymptotic equivalence of (2.1) and (2.2)

In this section we shall show that, under some conditions for a given solution $y = y(n)$ of (2.2), there exists a solution $x = x(n)$ of (2.1) such that (2.3) holds and conversely. The technique used in the proofs is a combination of the Schauder fixed point theorem and the theory of difference inequalities (Lemma 3.1).

THEOREM 4.1. *Suppose that the following conditions are satisfied:*

a) *there exists a nonsingular matrix* $\Delta(n)$ *such that*

$$(4.1) \qquad |\Delta(n) X(n)| \leq \alpha(n), \; n \in N(n_0),$$

where $\alpha(n)$ *is a positive function for* $n \in N(n_0)$;

b) *there exists a non-negative function* $a(n)$ *such that*

$$(4.2) \qquad |X^{-1}(n+1) f(n, y)| \leq a(n)\, \omega\left(\frac{|(\Delta(n) y|}{\alpha(n)}\right), \qquad \text{for} \quad n \in N(n_0), \; |y| < \infty,$$

where $\omega(r)$ *is a function having the properties from Lemma* 3.2 *and, in addition,*

$$(4.3) \qquad \Omega(z) \to \infty, \qquad \text{for} \quad z \to \infty;$$

c) *there exists a non-negative function* $h(n, s)$ *defined for* $n, s \in N(n_0)$, $s \leq n$, *such that*

$$(4.4) \qquad |X^{-1}(n+1) k(n, s, y)| \leq h(n, s)\, \omega\left(\frac{|\Delta(n) y|}{\alpha(n)}\right), \; n, s \in N(n_0), \; s \leq n;$$

d) *the functions* $a(n)$ *and* $h(n, s)$ *satisfy*

$$(4.5) \qquad \sum_{s=n_0}^{\infty}\left[a(s) + \sum_{1=n_0}^{s} h(s, 1)\right] < \infty,$$

Then, to each solution $y = y(n)$ *of* (2.1) *there exists a solution* $x = x(n)$ *of* (2.1) *such that*

$$|\Delta(n)[x(n) - y(n)]| = o(\alpha(n)), \qquad n \to \infty. \tag{4.6}$$

Proof. To prove the existence of $x(n)$ satisfying (4.6) is the same as to prove the existence of a constant vector c such that (4.6) holds with $x(n) = X(n)c$. Let $y = y(n)$ be a solution of (2.2) satisfying $y(n_0) = y_0$. Then,

$$y(n) = X(n)y_0 + X(n)\sum_{s=n_0}^{n-1} X^{-1}(s+1)\left[f(s, y(s)) + \sum_{1=n_0}^{s} k(s, 1, y(1))\right], \qquad n \in N(n_0), \tag{4.7}$$

from which

$$\frac{|\Delta(n)y(n)|}{\alpha(n)} \le |y_0| + \sum_{s=n_0}^{n-1} a(s)\,\omega\left(\frac{|\Delta(s)y(s)|}{\alpha(s)}\right) + \sum_{1=n_0}^{s} h(s, 1)\,\omega\left(\frac{|\Delta(1)y(1)|}{\alpha(1)}\right).$$

Applying Lemma 3.1 one obtains

$$\Omega\left(\frac{|\Delta(n)y(n)|}{\alpha(n)}\right) \le \Omega(|y_0|) + \sum_{s=n_0}^{n-1}\left[a(s) + \sum_{1=n_0}^{s} h(s, 1)\right]$$

and according to (4.3) and (4.5) we have $\frac{|\Delta(n)y(n)|}{\alpha(n)} \le M$, for $n \in N(n_0)$, where M is a positive constant. From

$$\left|\sum_{s=n_0}^{n-1} X^{-1}(s+1)\left[f(s, y(s)) + \sum_{1=n_0}^{s} k(s, 1, y(1))\right]\right| \le$$

$$\le \sum_{s=n_0}^{n-1}\left[a(s)\,\omega\left(\frac{|\Delta(s)y(s)|}{\alpha(s)}\right) + \sum_{1=n_0}^{s} h(s, 1)\,\omega\left(\frac{|\Delta(1)y(1)|}{\alpha(1)}\right)\right] \le$$

$$\le \omega(M)\sum_{s=n_0}^{n-1}\left[a(s) + \sum_{1=n_0}^{s} h(s, 1)\right], \qquad n \in N(n_0),$$

it follows that there exists

$$c = \lim_{n\to\infty}\sum_{s=n_0}^{n-1} X^{-1}(s+1)\left[f(s, y(s)) + \sum_{1=n_0}^{s} k(s, 1, y(1))\right] + y_0. \tag{4.8}$$

Using (4.8), we may write (4.7) as

$$\Delta(n)y(n) = \Delta(n)X(n)y_0 +$$

$$+ \Delta(n)X(n)\sum_{s=n_0}^{\infty} X^{-1}(s+1)\left[f(s, y(s)) + \sum_{1=n_0}^{s} k(s, 1, y(1))\right] -$$

$$-\Delta(n)\,X(n)\sum_{s=n}^{\infty}X^{-1}(s+1)\left[f(s\,,y(s))+\sum_{1=n_0}^{s}k(s\,,1\,,y(1))\right]=$$

$$=\Delta(n)\,X(n)\,c-$$

$$-\Delta(n)\,X(n)\sum_{s=n}^{\infty}X^{-1}(s+1)\left[f(s\,,y(s))+\sum_{1=n_0}^{s}k(s\,,1\,,y(1))\right].$$

Thus, from (4.1) and (4.5) it follows

$$\frac{|\Delta(n)\,[y(n)-X(n)\,c]\,|}{\alpha(n)}\leq\sum_{s=n}^{\infty}\left[a(s)+\sum_{1=n_0}^{s}h(s\,,1)\right]\to 0\quad\text{as}\quad n\to\infty\,,$$

i.e. the order relation (4.6).

Theorem 4.2 below, deals with a converse problem to that considered in Theorem 4.1 above. We note that this converse theorem holds only with an additional condition concerning the constant c from $x(n)=X(n)\,c$.

THEOREM 4.2. *Let the hypotheses of Theorem* 4.1 *be satisfied. Then, given any solution* $x(n)=X(n)\,c$ *of* (2.1) *with* $|c|<M$, *where* $M=\sum_{s=n}^{\infty}\left[a(s)+\sum_{1=n_0}^{s}h(s\,,1)\right]$, *there exists a solution* $y=y(n)$ *of* (2.2) *such that* (4.6) *holds.*

Proof. Let c satisfy $|c|<M$. We define $\eta=(M-|c|)/2$ and choose $n_1\in N(n_0)$ large enough so that

$$\sum_{s=n_1}^{\infty}\left[a(s)+\sum_{1=n_0}^{s}h(s\,,1)\right]<\eta/\omega(M)\,.$$

We shall establish the existence of a solution of the equation

$$y(n)=X(n)\,c-\tag{4.9}$$

$$-X(n)\sum_{s=n}^{\infty}X^{-1}(s+1)\left[f(s\,,y(s))+\sum_{1=n_0}^{s}k(s\,,1\,,y(1))\right],\qquad n\in N(n_1)\,.$$

Consider the set

$A=\{u\,;u(n)=\Delta(n)\,y(n)/\alpha(n),$ where $y(n)$ is defined on $N(n_1)$ and $|u|\leq M-\eta\}$, and define the operator

$$\tau u(n)=\frac{\Delta(n)\,X(n)\,c}{\alpha(n)}-\frac{\Delta(n)\,X(n)}{\alpha(n)}\sum_{s=n}^{\infty}X^{-1}(s+1)\,\cdot\tag{4.10}$$

$$\cdot\left[f(s\,,\Delta^{-1}(s)\,\alpha(s)\,u(s))+\sum_{1=n_0}^{s}k(s\,,1\,,\Delta^{-1}(1)\,\alpha(1)\,u(1))\right],\qquad n\in N(n_1)\,.$$

From (4.1), (4.2) and (4.4) for $u(n) \in A$, we have

$$|\tau u(n)| \leq |c| + \omega(M) \sum_{s=n}^{\infty} \left[a(s) + \sum_{1=n_0}^{s} h(s, 1) \right] \leq |c| + \eta = M - \eta, \quad n \in N(n_1),$$

that is, $\tau(A) \subset A$. Next, it will shown that τ is continuous.

Consider the sequence $\{u_i(n)\}$, $u_i(n) \in A$, uniformly convergent on $N_m(n_1)$ to $u(n) \in A$. Let $\varepsilon > 0$ and choose $n_2 \in N(n_1)$ so large that

$$(4.11) \qquad \sum_{s=n_2}^{\infty} \left[a(s) + \sum_{1=n_0}^{s} h(s, 1) \right] < \varepsilon/4\, \omega(M).$$

Then, using (4.10) we get

$$\begin{aligned} |\tau u(n) - \tau u_i(n)| &\leq \sum_{s=n}^{n_2-1} |X^{-1}(s+1)| \, [|f(s, \Delta^{-1}(s)\, \alpha(s)\, u(s)) - \\ &- f(s, \Delta^{-1}(s)\, \alpha(s)\, u_i(s))| + \sum_{1=n_0}^{s} |k(s, 1, \Delta^{-1}(1)\, \alpha(1)\, u(1)) - \\ &- k(s, 1, \Delta^{-1}(1)\, \alpha(1)\, u_i(1))|] + \\ &+ \sum_{s=n_2}^{\infty} \left\{ \left| X^{-1}(s+1) \left[f(s, \Delta^{-1}(s)\, \alpha(s)\, u(s)) + \right. \right. \right. \\ &\left. \left. + \sum_{1=n_0}^{s} k(s, 1, \Delta^{-1}(1)\, \alpha(1)\, u_i(1)) \right] \right| + \\ &+ \left| X^{-1}(s+1) \left[f(s, \Delta^{-1}(s)\, \alpha(s)\, u_i(s)) + \right. \right. \\ &\left. \left. \left. + \sum_{1=n_0}^{s} k(s, 1, \Delta^{-1}(1)\, \alpha(1)\, u_i(1)) \right] \right| \right\}. \end{aligned}$$

Using (4.2), (4.4) and (4.11) it follows that the second term (the sum from n_2 to ∞) is dominated by $[\omega(M)\, \varepsilon/2\, \omega(M)] = \varepsilon/2$. By the uniform convergence of $\{u_i(n)\}$ on every $N_m(n_1)$, there exists $n_3 = n_3(\varepsilon, n_2)$ such that for $i \in N(n_3)$

$$|f(s, \Delta^{-1}(s)\, \alpha(s)\, u(s)) - f(s, \Delta^{-1}(s)\, \alpha(s)\, u_i(s))| < \\ < \varepsilon/4\, K(1+n_2)(n_2-n_1)$$

and

$$|k(n, s, \Delta^{-1}(s)\, \alpha(s)\, u(s)) - k(n, s, \Delta^{-1}(s)\, \alpha(s)\, u_i(s))| < \\ < \varepsilon/4\, K(1+n_2)(n_2-n_1),$$

where

$$K = \max\{|X^{-1}(n)|; \quad n \in N_{n_2}(n_1)\}.$$

Hence, $|\tau u(n) - \tau u_i(n)| < \varepsilon$ for $n \in N_{n_2}(n_1)$ and $i \in N(n_3)$, i.e., the sequence $\{u_i(n)\}$ is uniformly convergent to $\tau u(n)$ on every set $N_m(n_1)$ and therefore τ is a continuous operator. Since $\tau(A) \subset A$, then $\tau(A)$ is uniformly bounded. The equicontinuity of the family $\tau(A)$ follows because the functions in $\tau(A)$ are defined for a discrete variable n. By the Schauder's theorem, there exists a solution of equation (4.9) which is a solution of equation (2.2) and satisfies (4.6). The proof of the theorem is complete.

Remark. Interesting particular cases may be obtained if we choose $\Delta(n)$, $\alpha(n)$ and $\omega(r)$ of particular forms [3].

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