Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Sangappa Mallappa Sarangi, Basappa Amrutappa Uralegaddi

The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients, I

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **65** (1978), n.1-2, p. 38–42.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1978_8_65_1-2_38_0>

Funzioni di variabile complessa. — *The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients*, I. Nota (*) di SANGAPPA MALLAPPA SARANGI e BASAPPA AMRUTAPPA URALEGADDI, presentata dal Socio G. SANSONE.

RIASSUNTO. — Per le funzioni della forma $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ con $\operatorname{Re}(f(z)/z) > \alpha$, $\operatorname{Re}(f'(z)) > \alpha$ si determinano il raggio di convessità e il raggio di stellarità.

1. INTRODUCTION

In this paper we determine coefficient estimates and comparable results for functions of the form $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, which satisfy $\operatorname{Re}(f(z)/z) > \alpha$ and $\operatorname{Re} f'(z) > \alpha$ for $|z| < 1$. Also, we find the radius of convexity for functions $f(z)$ which are analytic and satisfy $\operatorname{Re} f'(z) > \alpha$ for $|z| < 1$, and the radius of starlikeness for functions $f(z)$ which are analytic and satisfy $\operatorname{Re}(f(z)/z) > \alpha$ for $|z| < 1$. We consider the problem of finding the radius of starlikeness for functions $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ which are analytic and satisfy $\operatorname{Re}(f(z)/g(z)) > 0$ for $|z| < 1$, where $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ is analytic and univalent for $|z| < 1$. The problem is solved in the case that $g(z)$ is either starlike of order α or convex of order α.

The standard notations will be used for the classes of functions studied. The function $f(z)$ is said to be starlike of order α for $0 \leq \alpha < 1$, if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$ for $|z| < 1$, and is said to be convex of order α if $\operatorname{Re}(zf''(z)/f'(z) + 1) > \alpha$ for $|z| < 1$.

In [5] Herb Silverman has examined the class of univalent functions with negative coefficients. We shall employ similar techniques.

2. THEOREM 1. *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$. *Then*

(i) $f(z)$ *has* $\operatorname{Re}(f(z)/z) > \alpha$ *for* $|z| < 1$, *iff* $\sum_{n=2}^{\infty} |a_n| \leq 1 - \alpha$.

(ii) $f(z)$ *has* $\operatorname{Re} f'(z) > \alpha$ *for* $|z| < 1$ *iff* $\sum_{n=2}^{\infty} n |a_n| \leq 1 - \alpha$.

(*) Pervenuta all'Accademia il 4 agosto 1978.

Proof. (i) suppose

$$(1)\qquad \sum_{n=2}^{\infty} |a_n| \leq 1 - \alpha .$$

It is sufficient to show that $f(z)/z$ lies in a circle with centre at $w = 1$ and radius $1 - \alpha$, we have

$$|(f(z)/z) - 1| = \left| \sum_{n=2}^{\infty} |a_n| z^{n-1} \right| \leq \sum_{n=2}^{\infty} |a_n| .$$

This last expression is bounded above by $1 - \alpha$ if (1) is satisfied.

Conversely suppose that

$$\operatorname{Re}(f(z)/z) = \operatorname{Re}\left\{1 - \sum_{n=2}^{\infty} |a_n| z^{n-1}\right\} > \alpha .$$

Choose values of z on the real axis so that $f(z)/z$ is real. Letting $z \to 1$ along the real axis we obtain (1).

(ii) The proof is similar to that of (i).

COROLLARY. *For functions* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, *we have*

(i) *If* $f(z)$ *has* $\operatorname{Re}(f(z)/z) > \alpha$ *for* $|z| < 1$, *then* $|a_n| \leq 1 - \alpha$ *with equality for functions* $f(z) = z - (1 - \alpha) z^n$.

(ii) *If* $f(z)$ *has* $\operatorname{Re} f'(z) > \alpha$ *for* $|z| < 1$ *then* $|a_n| \leq \frac{1 - \alpha}{n}$ *with equality for functions* $f(z) = z - \frac{(1 - \alpha)}{n} z^n$.

THEOREM 2. *If* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *has* $\operatorname{Re} f'(z) > \alpha$ *for* $|z| < 1$, *then* $\operatorname{Re}(f(z)/z) > \frac{1 + \alpha}{2}$ *for* $|z| < 1$.

Proof. In view of Theorem 1, we have to prove that,

$$\sum_{n=2}^{\infty} \frac{n |a_n|}{1 - \alpha} \leq 1 \Rightarrow \sum_{n=2}^{\infty} \frac{2 |a_n|}{1 - \alpha} \leq 1 .$$

It is sufficient to show that

$$\frac{2 |a_n|}{1 - \alpha} \leq \frac{n |a_n|}{1 - \alpha} \qquad \text{for} \quad n = 2, 3, 4, \cdots .$$

The result follows.

The estimate is sharp for the function $f(z) = z - \frac{(1 - \alpha)}{2} z^2$.

THEOREM 3. *If* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *has* $\operatorname{Re}(f(z)/z) > \alpha$ *for* $|z| < 1$, *then* $\operatorname{Re} f'(z) > 0$ *in the disk*

$$|z| < r = r(\alpha) = \inf_n \left(\frac{1}{n(1-\alpha)}\right)^{1/(n-1)} \qquad n = 2, 3, 4, \cdots.$$

Proof. It is sufficient to show that

$$|f'(z) - 1| \leq 1 \qquad \text{for} \quad |z| \leq r(\alpha).$$

We have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n\,|a_n|\,|z|^{n-1}.$$

Hence $|f'(z) - 1| \leq 1$ if

$$\sum_{n=2}^{\infty} n\,|a_n|\,|z|^{n-1} \leq 1. \tag{2}$$

From Theorem 1, $\sum_{n=2}^{\infty} \frac{|a_n|}{1-\alpha} \leq 1$. Hence (2) will be satisfied if

$$n\,|a_n|\,|z|^{n-1} \leq \frac{|a_n|}{1-\alpha}, \qquad n = 2, 3, 4, \cdots.$$

Solving this for $|z|$, we obtain

$$|z| \leq \left(\frac{1}{n(1-\alpha)}\right)^{1/(n-1)}, \qquad n = 2, 3, 4, \cdots. \tag{3}$$

Writing $|z| = r(\alpha)$ in (3) the result follows.

The estimate is sharp for the function $f(z) = z - (1-\alpha) z^n$ for some n. We have $r(0) = 1/2$ and $r(1/2) = 1/\sqrt[3]{2}$.

THEOREM 4. *If* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *is analytic and has* $\operatorname{Re} f'(z) > \alpha$ *for* $|z| < 1$, *then* $f(z)$ *is convex for*

$$|z| < r = r(\alpha) = \inf_n \left(\frac{1}{n(1-\alpha)}\right)^{1/(n-1)}, \qquad n = 2, 3, 4, \cdots.$$

Proof. It is sufficient to show that $|zf''(z)/f'(z)| \leq 1$ for $|z| < r(\alpha)$. We have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)\,|a_n|\,|z|^{n-1}}{1 - \sum_{n=2}^{\infty} n\,|a_n|\,|z|^{n-1}}.$$

Hence $|zf''(z)/f'(z)| \leq 1$ if

$$\sum_{n=2}^{\infty} n(n-1)\,|a_n|\,|z|^{n-1} \leq 1 - \sum_{n=2}^{\infty} n\,|a_n|\,|z|^{n-1}. \tag{4}$$

This reduces to

$$\sum_{n=2}^{\infty} n^2\,|a_n|\,|z|^{n-1} \leq 1. \tag{5}$$

The remaining part of the proof is similar to that of Theorem 4. The estimate is sharp for the function $f(z) = z - \dfrac{1-\alpha}{n} z^n$ for some n.

THEOREM 5. *If* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *has* $\operatorname{Re} f(z)/z > \alpha$, *then* $f(z)$ *is univalent and starlike in the disk*

$$|z| < r = r(\alpha) = \inf_n \left(\frac{1}{n(1-\alpha)}\right)^{1/n-1} \qquad \textit{for} \quad n = 2, 3, 4, \cdots.$$

The proof is omitted. The estimate is sharp for the function $f(z) = z - (1-\alpha) z^n$ for some n. Again we have $r(0) = 1/2$ and $r(1/2) = 1/\sqrt[3]{2}$. Results comparable to Theorem 5 are known for a wider class of functions [1. Theorem 3] and [2. Theorem 2].

LEMMA. *Let* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *be analytic for* $|z| < 1$ *and* $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ *be analytic and univalent on* $\{z : |z| < 1\}$. *If* $\operatorname{Re}(f(z)/g(z)) > \alpha$ *for* $|z| < 1$ *then* $\sum_{n=2}^{\infty} |a_n| - \alpha \sum_{n=2}^{\infty} |b_n| \leq 1 - \alpha$.

Proof. We have

$$\operatorname{Re} \frac{f(z)}{g(z)} = \operatorname{Re} \frac{1 - \sum_{n=2}^{\infty} |a_n| z^{n-1}}{1 - \sum_{n=2}^{\infty} |b_n| z^{n-1}}.$$

Choose values of z on the real axis so that $f(z)/g(z)$ is real. Since $\operatorname{Re} f(z)/g(z) > \alpha$, we have

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} > \alpha \left(1 - \sum_{n=2}^{\infty} |b_n| z^{n-1}\right). \tag{6}$$

Let $z \to 1$ along the real axis. Inequality (6) reduces to

$$\sum_{n=2}^{\infty} |a_n| - \alpha \sum_{n=2}^{\infty} |b_n| \leq 1 - \alpha.$$

THEOREM 6. *Suppose that* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *is analytic for* $|z| < 1$ *and* $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ *is analytic and univalent on* $\{z : |z| < 1\}$. *If* $\mathrm{Re}\,(f(z)/g(z)) > 0$ *for* $|z| < 1$ *then* $f(z)$ *is univalent and starlike in* $|z| < 1/2$.

Proof. Since $\mathrm{Re}\, f(z)/g(z) > 0$, from the lemma we get $\sum_{n=2}^{\infty} |a_n| \leq 1$. Hence from Theorem 1, $\mathrm{Re}\, f(z)/z > 0$ and applying Theorem 5 we get the result. The result is sharp for the functions $f(z) = z - z^2$ where $g(z) = z - \frac{1-\alpha}{2-\alpha} z^2$ for $0 \leq \alpha \leq 1$. Since $f'(z) = 0$ at $z = 1/2$, $f(z)$ is not univalent in $|z| < r$ if $r > 1/2$. Since the functions $g(z) = z - \frac{1-\alpha}{2-\alpha} z^2$ are starlike of order α, Theorem 6 is comparable to the following sharp result of J. S. Ratti [4]. If $f(z) = z + a_2 z^2 + \cdots$ and $g(z) = z + b_2 z^2 + \cdots$ are analytic for $|z| < 1$ and $g(z)$ is starlike of order α for $|z| < 1$ and if $\mathrm{Re}\, f(z)/g(z) > 0$ for $|z| < 1$, then $f(z)$ is univalent and starlike for $|z| < r$ where

$$r = \frac{(\alpha^2 - 2\alpha + 3)^{1/2} + \alpha - 2}{2\alpha - 1} \quad \text{provided} \quad \alpha \neq 1/2 \quad \text{and} \quad r = 1/3$$

when $\alpha = 1/2$. Also the following comparable result has been shown by MacGregor [3]. If $f(z) = z + a_2 z^2 + \cdots$ and $g(z) = z + b_2 z^2 + \cdots$ are analytic for $|z| < 1$ and $g(z)$ is univalent in $|z| < 1$ and if $\mathrm{Re}\,(f(z)/g(z)) > 0$ for $|z| < 1$ then $f(z)$ si univalent in $|z| < 1/5$.

REFERENCES

[1] T. H. MACGREGOR (1962) – *Functions whose derivative has a positive real part*, «Trans. Amer. Math. Soc.», *104*, 532–537.

[2] T. H. MACGREGOR (1963) – *The radius of convexity for starlike functions of order 1/2*, «Proc. Amer. Math. Soc.», *14*, 71–76.

[3] T. H. MACGREGOR (1963) – *The radius of univalence of certain analytic functions*, «Proc. Amer. Math. Soc.», *14*, 514–520.

[4] J. S. RATTI (1968) – *The radius of univalence of certain analytic functions*, «Math. Z.», *107*, 241–248.

[5] HERB SILVERMAN (1975) – *Univalent functions with negative coefficients*, «Proc. Amer. Math. Soc.», *51*, 109–116.