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On A-p-summing operators

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Analisi matematica. — *On* A-*p-summing operators.* Nota di NICOLAE TIȚA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si generalizzano alcune proprietà degli operatori A-p sommabili e si risolve una questione presentata da altri A.

In this paper the class of A-p-summing operators [2], [1] is generalized and an affirmative answer to the problem raised in [2] is presented.

Let Φ be a norm function of R. Schatten [4], [3]

$$(\Phi : \hat{c} \rightarrow R_+ ; \Phi(x+y) \leq \Phi(x) + \Phi(y) , x , y \in \hat{c} ;$$

$$\Phi(\alpha x) = |\alpha| \Phi(x) , \alpha \in R , x \in \hat{c} ; \Phi(1, 0, 0, \cdots) = 1 ;$$

$$\Phi(x_1, x_2, \cdots, x_n, 0, 0, \cdots) = \Phi(|x_{i_1}|, |x_{i_2}|, \cdots, |x_{i_n}|, 0, 0, \cdots).$$

Here $i_1, i_2, \cdots, i_n$ is a permutation of $1, 2, \cdots, n$ and $\hat{c}$ is the spaces of all sequences $x = (x_1, x_2, \cdots, x_n, 0, 0, \cdots)$, $x_i \in R$, $n < \infty$.

Let $\hat{k}$ be the space of the sequences $x \in \hat{c}$ for which $x_i \geq 0$. In [3] it is generalized the notion of the dual function from [4], in the following way.

$$\Phi^*_\Psi(x) = \sup_{y \in \hat{k}} \frac{\Psi(xy)}{\Phi(y)} \quad ; \quad x \cdot y = \{x_1 y_1, x_2 y_2, \cdots, x_n y_n, \cdots\} . \tag{1}$$

(In the particular case of $\Phi_p(x) = (\Sigma x_i^p)^{1/p}$, $p > 1$; $\Phi_r(x) = (\Sigma x_i^r)^{1/r}$, $r > 1$, $x_i \geq 0$, one has $(\Phi^*_r)_{\Phi_p} \equiv \Phi_q$, where $\Phi_q(x) = (\Sigma x_i^q)^{1/q}$ and $1/p = 1/r + 1/q$).

Let $A = \| a_{ij} \|$ be an infinite matrix satisfying some properties [1], [2].

(*) Nella seduta del 15 giugno 1978.

DEFINITION 1. *Let* E, F *be normed spaces. A linear operator* $T : E \to F$ *is called an* A-Φ-*summing operator if for all sequences* $\{x_n\} \in E$ (1) *there exists a constant* $c > 0$ *such that*

$$\Phi\left(\left\{\sum_n a_{mn} \| Tx_n \|\right\}\right) = c \sup_{\|a\| \le 1} \Phi\left(\left\{\sum_n a_{mn} | \langle x_n , a \rangle |\right\}\right), \; a \in E'.$$

The smallest constant c such that [2] holds is denoted by $\Pi_\phi(T)$.

Remark. For $\Phi_p(x) = (\Sigma x_i^p)^{1/p}$ one has the class of A-p-summing operators [1], [2] and if $A = \| \delta_{ij} \|$ ($\delta_{ii} = 1$; $\delta_{ij} = 0$, $i \neq j$) one has the class of Φ-summing operators [5].

Let $\Pi_\Phi(E, F)$ be the set of all A-ϕ-summing operators from E to F.

PROPOSITION 1. $\Pi_\Phi(E, F)$ *is a normed space with respect to* $\Pi_\Phi(T)$.

Proof. Let $S, T \in \Pi_\Phi(E, F)$. Then for any sequence $\{x_n\} \in E$

$$\Phi\left(\left\{\sum_n a_{mn} \| (T + S) x_n \|\right\}\right) \le \Phi\left(\left\{\sum_n a_{mn} \| Tx_n \|\right\}\right) +$$

$$+ \Phi\left(\left\{\sum_n a_{mn} \| Sx_n \|\right\}\right) \le \Pi_\phi(T) \sup_{\|a\| \le 1} \Phi\left(\left\{\sum_n a_{mn} | \langle x_n , a \rangle |\right\}\right) +$$

$$+ \Pi_\phi(S) \sup_{\|a\| \le 1} \Phi(\Sigma a_{mn} | \langle x_n , a \rangle |) = [\Pi_\phi(T) + \Pi_\Phi(S)] \sup_{\|a\| \le 1} \Phi(\Sigma a_{mn} | \langle x_n , a \rangle |).$$

Hence $T + S \in \Pi_\Phi(E, F)$ and $\Pi_\phi(T + S) \le \Pi_\phi(T) + \Pi_\phi(S)$. For any real α one has $\alpha T \in \Pi_\phi(E, F)$ if $T \in \Pi_\Phi(E, F)$ since $\Pi_\phi(\alpha T) = |\alpha| \, \Pi_\phi(T)$ (2).

PROPOSITION 2. *If* $T \in \Pi_\phi(E, F)$ *then* T *is bounded* ($T \in L(E, F)$).

Proof. Let $\{x_n\} = \{x, 0, 0, \cdots\}$. Then

$$\| Tx \| \, \Phi(\{a_{m1}\}) \le \Pi_\phi(T) \sup_{\|a\| \le 1} \Phi(\{a_{m1}\}) \cdot | \langle x , a \rangle | .$$

Hence

$$\| T \| \le \Pi_\phi(T) .$$

PROPOSITION 3. *If* F *is a Banach space then* $\Pi_\phi(E, F)$ *is a Banach space.*

Proof. Let $\{T_n\}$ be a Cauchy sequence in $\Pi_\phi(E, F)$ ($\lim_{n,m} \Pi_\Phi(T_n - T_m) = 0$).

(1) The sequence $\{x_n\}$ is a sequence of finite rank $\{x_n\}_{n=1}^k$, $k < \infty$, ($x_n = 0$ if $n > k$). $\Phi(x) = \sup_n \phi(x_1, x_2, \cdots, x_n, 0, 0, \cdots)$ if $x \notin \hat{c}$.

(2) $\Pi_\Phi(T) \ge 0$, $\Pi_\Phi(T) = 0$ iff $T = 0$.

Then $\{T_n\}$ is a Cauchy sequence in $L(E,F)$ since $\|T_n - T_m\| \leq \Pi_\phi(T_n - T_m)$. Since $L(E, F)$ is a Banach space one has $\lim\limits_n T_n = T \in L(E, F)$.

From the relation $\Pi_\phi(T_n - T_m) \langle \varepsilon, n, m \rangle N(\varepsilon)$ $(\forall \varepsilon > 0)$ and from the continuity of the functions Φ, one obtains

$$\lim_n \Pi_\Phi(T_n - T_m) = \Pi_\phi(T - T_m) \langle \varepsilon, m \rangle N(\varepsilon)) .$$

Hence $\lim\limits_n T_n = T$ (in the topology of $\Pi_\phi(E, F)$) and $T \in \Pi_\phi(E, F)$.

PROPOSITION 4. *Let* E, F, G *be normed spaces. If* $T \in L(E, F)$ *and* $S \in \Pi_\phi(F, G)$ then $ST \in \Pi_\Phi(E, G)$ and $\Pi_\Phi(ST) \leq \|T\| \Pi_\Phi(S)$. If $T \in \Pi_\Phi(E, F)$ and $S \in L(F, G)$ then $ST \in \Pi_\Phi(E, G)$ and $\Pi_\Phi(ST) \leq \|S\| \Pi_\Phi(T)$.

Proof. If $\{x_n\}$ is a sequence in E one has

$$\Phi\left(\left\{\sum_n a_{mn} \| STx_n \|\right\}\right) \leq \Pi_\Phi(S) \sup_{\|a\| \leq 1} \Phi\left(\left\{\sum_n a_{mn} | \langle Tx_n, a \rangle |\right\}\right) \leq$$

$$\leq \|T\| \cdot \Pi_\Phi(S) \sup_{\|a\| \leq 1} \Phi\left(\left\{\sum_n a_{mn} \left| \left\langle x_n, \frac{T^* a}{\|T\|} \right\rangle \right| \right\}\right) \leq$$

$$\leq \|T\| \cdot \Pi_\phi(S) \sup_{\|\bar{a}\| \leq 1} \Phi\left(\left\{\sum_n a_{mn} | \langle x_n, \bar{a} | \right\}\right)$$

$\left(\bar{a} = \dfrac{T^* a}{\|T\|} \text{ and } T^* \text{ is the conjugate operator to } T\right)$. Hence $ST \in \Pi_\Phi(E, G)$ and $\Pi_\phi(ST) \leq \|T\| \cdot \Pi_\phi(S)$.

The second part is straightforward.

PROPOSITION 5. *Let* ϕ, Ψ *be norm functions* $\Psi \neq \Psi_1, \Psi_\infty$ [3], *then if* $T \in \Pi_\Phi(E, F)$ one has $T \in \Pi_{\psi_\Phi^*}(E, F)$.

Proof. Consider the sequence $\{t_i x_j\} \in E$, $(t_i) \in \hat{k}$. Then

$$\Phi\left(\left\{\sum_j a_{ij} \| Tt_i x_j \|\right\}\right) \leq \Pi_\Phi(T) \sup_{\|a\| \leq 1} \Phi\left\{\left(\sum_j a_{ij} | \langle t_i x_j, a \rangle |\right\}\right).$$

Hence

$$\Phi\left(t_i \left\{\sum_j a_{ij} \| Tx_j \|\right\}\right) \leq \Pi_\phi(T) \sup_{\|a\| \leq 1} \Phi\left(t_i \left\{\sum_j a_{ij} | \langle x_j, a \rangle |\right\}\right) \leq$$

$$\leq \Pi_\phi(T) \sup_{\|a\| \leq 1} \Psi(\{t_i\}) \cdot \Psi_\Phi^*\left(\left\{\sum_j a_{ij} | \langle x_j, a \rangle |\right\}\right).$$

(3) $\Psi_1(x) = \Sigma |x_i|$, $\Psi_\infty(x) = \sup |x_n|$, $(x = \{x_1, \cdots, x_n, 0, 0, \cdots\})$.

Or

$$\frac{\Phi\left(t_i\left\{\sum_j a_{ij}\|\mathrm{T}x_j\|\right\}\right)}{\Psi(\{t_i\})} \leq \Pi_\Phi(\mathrm{T}) \sup_{\|a\|\leq 1} \Psi_\Phi^*(\{\Sigma a_{ij} \mid \langle x_j , a\rangle \mid\}) .$$

From the definition of the function Ψ_Φ^* one obtains

$$\Psi_\Phi^*\left(\left\{\sum_j a_{ij}\|\mathrm{T}x_j\|\right\}\right) \leq \Pi_\Phi(\mathrm{T}) \sup_{\|a\|\leq 1} \Psi_\Phi^*(\{\Sigma a_{ij} \mid \langle x_j , a\rangle \mid\}) .$$

Hence $\mathrm{T} \in \Pi_{\Psi_\Phi^*}(\mathrm{E}, \mathrm{F})$.

Remark. If $\Phi \equiv \Phi_p$ and $\Psi \equiv \Psi_r\,(1 < r < \infty)$ one has $\Psi_\Phi^* = \Phi_q$ if $1/p = 1/r + 1/q$. Since $1 < r < \infty$ one has $q > p$ and hence if $\mathrm{T} \in \Pi_p$ then $\mathrm{T} \in \Pi_q$. This is an affirmative answer to the problem raised in [2].

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