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A hysteretic semilinear parabolic equation

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Analisi matematica. — *A hysteretic semilinear parabolic equation.* Nota (*) di EUGENIO SINESTRARI e PAOLA VERNOLE, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si studia il problema di Cauchy per una equazione parabolica semilineare con ritardo. Si dimostra l'esistenza e l'unicità della soluzione e la dipendenza continua da tutti i dati. Inoltre si studia la regolarità e l'esistenza in grande.

1. INTRODUCTION

In this paper we study the problem of finding a function u from $R_+ \times \overline{\Omega}$ into R, solution of the following problem:

$$(1) \quad \begin{cases} u_t(t,x) = \Delta u(t,x) + \\ \qquad + \int_{-r}^{t} H(t,s,x,u(t,x),u(s,x),\nabla_x u(t,x),\nabla_x u(s,x))\, ds \\ t \geq 0, x \in \Omega \; ; \; u(t,x) = 0, t \geq 0, x \in \partial\Omega \; ; \\ \qquad u(t,x) = h(t,x), -r \leq t \leq 0, x \in \Omega . \end{cases}$$

Here $\Omega \subseteq R^n$ is a bounded open set with 'nice' boundary $\partial\Omega$, r is a positive number and h is a given function from $[-r, 0] \times \overline{\Omega}$ to R. We shall assume that $H(t, s, x, u_1, u_2, w_1, w_2)$ is a continuously differentiable function from $A = \{(t, s, x, u_1, u_2, w_1, w_2), -r \leq s \leq t < +\infty \; ; u_1, u_2 \in R \; ; w_1, w_2 \in R^n)\}$ into R.

We shall write the equation (1) as an abstract retarded functional differential equation in an interpolation space and we will use some abstract theorems to solve it they; extend the results of [4] to our problem.

2. EXISTENCE, UNIQUENESS AND APPROXIMATION OF THE SOLUTIONS

Let us choose θ and p such that

$$(2) \qquad 1/2 < \theta < 1 \quad , \quad n/(2\theta - 1) < p < +\infty .$$

Let $E = L^p(\Omega)$. It is known that the operator Λ defined by: $D_\Lambda = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $\Lambda u = \Delta u$ verifies:

(L_1) Λ is the infinitesimal generator of a holomorphic semigroup $\{e^{\Lambda t}\}$ on E and $\| e^{\Lambda t} \|_{\mathcal{L}(E)} \leq M_1$, $t \geq 0$, $\| \Lambda e^{\Lambda t} \|_{\mathcal{L}(E)} \leq M_2/t$, $t > 0$.

(*) Pervenuta all'Accademia il 26 luglio 1978.

Let us denote by $D_A(\theta, p)$ the real interpolation space between D_A and E, with parameters $1-\theta$ and p (see [3]): we have $D_A(\theta, p) = W_0^{2,p}(\Omega)$. $\|\cdot\|$ denotes the norm in E and $\|\cdot\|_{\theta,p}$ the norm in $D_A(\theta, p)$. If $a < b$ let us set $X_{a,b} = C^0([a, b], E)$ and $\tilde{X}_{a,b} = C^0([a, b], D_A(\theta, p))$. Let $u(t) = u(t, \cdot)$, $h(t) = h(t, \cdot)$ and

$$\mathscr{F}(t, u(s), -r \leq s \leq t)(x) = \tag{3}$$

$$= \int_{-1}^{t} H(t, s, x, u(t, x), u(s, x), \nabla_x u(t, x), \nabla_x u(s, x))\, ds .$$

Then problem (1) can be written as:

$$\begin{cases} u'(t) = Au(t) + \mathscr{F}(t, u(s), -r \leq s \leq t) & t \geq 0 \\ u(t) = h(t) & -r \leq t \leq 0 . \end{cases} \tag{4}$$

Here $u \in \tilde{X}_{-r,T}$, $h \in \tilde{X}_{-r,0}$ and $\mathscr{F}(t, u(s), -r \leq s \leq t)$ is a Volterra functional determined by $t > 0$ and the values of $u(s)$ for $-r \leq s \leq t$: following Driver [2] we call (4) a hysteretic parabolic equation and denote $\mathscr{F}(t, u(s), -r \leq s \leq t)$ by $\mathscr{F}(t, u(\cdot))$ for $t > 0$. We call a mild solution of the problem (1) or (4) in $[-r, T]$ a function $u \in \tilde{X}_{-r,T}$ satisfying:

$$\text{(P)} \quad \begin{cases} u(t) = e^{At} h(0) + \displaystyle\int_0^t e^{A(t-s)} \mathscr{F}(s, u(\cdot))\, ds & t \in [0, T], \\ u(t) = h(t) & t \in [-r, 0] . \end{cases}$$

It is easy to check (owing to (2) and to the continuous differentiability of H) that the following properties of $\mathscr{F}$ hold:

F_1) Given $T > 0$ and $u \in \tilde{X}_{-r,T}$, $t \to \mathscr{F}(t, u(\cdot))$ is continuous from $[0, T]$ to E.

F_2) Given $T > 0$ and $u_0 \in \tilde{X}_{-r,T}$ there exist positive numbers r_0 and k_0 such that $\sup\{\|\mathscr{F}(t, u_1(\cdot)) - \mathscr{F}(t, u_2(\cdot))\|, 0 \leq t \leq T\} \leq \leq k_0 \|u_1 - u_2\|_{\tilde{X}_{-r,T}}$ for each $u_1, u_2 \in \tilde{X}_{-r,T}$ such that $\|u_i(t) - - u_0(t)\|_{\theta,p} \leq r_0$, $i = 1, 2$, $t \in [-r, T]$.

THEOREM 1. *For each* $h \in \tilde{X}_{-r,0}$, *there is a positive number* T_0 *such that there exists a unique mild solution of* (1) *in* $[-r, T_0]$. *Moreover there exist* $\delta_0 > 0$ *such that if* $h_1 \in \tilde{X}_{-r,0}$ *and* $\|h_1 - h\|_{\tilde{X}_{-r,0}} \leq \delta_0$ *the mild solution* u_1 *corresponding to the initial value* h_1 *is still defined in* $[-r, T_0]$ *and we have* $\|u_1 - u\|_{\tilde{X}_{0,T_0}} \leq c_0 \|h - h_1\|_{\tilde{X}_{-r,0}}$.

The proof is based on a suitable use of the contraction mapping principle in order to obtain the above mentioned dependence of T_0 on the initial function h and can be carried out on the same lines of the proof of Theorem 2.1 of [4].

It follows from theorem 1 the uniqueness in the large of the solution of (P).

For every $n \in \mathrm{N}$ let $\Lambda_n : \mathrm{D} \subseteq \mathrm{E} \to \mathrm{E}$ be a linear operator verifying the condition (L_1) with the same constants M_1 and M_2. Let the following condition hold:

(A) $\lim_{n\to\infty} \| e^{\Lambda_n t} x - e^{\Lambda t} x \|_{\theta,p} = 0$ uniformly in $t \in \mathrm{I}$ (compact interval of R_+) and $x \in \mathrm{K}$ (compact set of E).

For every $n \in \mathrm{N}$ let $\mathcal{F}_n$ be a functional verifying F_1) and F_2) with the same constants r_0 and k_0. Let the following condition hold:

(B) Given $\mathrm{T} > 0$ and $u \in \tilde{\mathrm{X}}_{-r,\mathrm{T}}$ then $\lim_{n\to\infty} \sup \{\| \mathcal{F}_n(t, v)(\cdot)) - \mathcal{F}(t, v(\cdot)) \|, t \in [0, \mathrm{T}]\} = 0$.

The following theorem describes the dependence of the solution on all the data Λ, $\mathcal{F}$ and h: the proof is similar to that of Theorem 3.1 of [4].

THEOREM 2. *Given $h \in \tilde{\mathrm{X}}_{-r,0}$ let δ_0 and T_0 be the positive constants given by Theorem 1 and let (A) and (B) hold. If we choose $h_n \in \tilde{\mathrm{X}}_{-r,0}$, $\| h_n - h \|_{\tilde{\mathrm{X}}_{-r,0}} \leq \delta_0$ $(n \in \mathrm{N})$ and $\mathrm{T}_0^* \in]0, \mathrm{T}_0[$, then for each $n \geq n_0$ (depending on h and T_0^*) there exists a unique solution u_n to the following problem:*

$$
(\mathrm{P}_n) \quad \begin{cases} u_n(t) = e^{\Lambda_n t} h_n(0) + \displaystyle\int_0^t e^{\Lambda_n(t-s)} \mathcal{F}_n(s, u_n(\cdot))\, \mathrm{d}s \,, & t \in [0, \mathrm{T}_0^*] \,, \\ u_n(t) = h_n(t) \,, & t \in [-r, 0] \,. \end{cases}
$$

If in addition u is the solution to (P) *and $\lim_{n\to\infty} h_n = h$ in $\tilde{\mathrm{X}}_{-r,0}$ then we have $\lim_{n\to\infty} u_n = u$ in $\tilde{\mathrm{X}}_{-r,\mathrm{T}_0^*}$.*

From this theorem and by means of a compactness argument we can derive a result which shows that a solution of (P) can be uniformly approximated by solutions of (P_n) in every compact interval of R_+.

THEOREM 3. *Let Λ_n, $\mathcal{F}_n$ and h_n verify the assumptions of Theorem 2. Let $u : [-r, \mathrm{T}] \to \mathrm{D}_\Lambda(\theta, p)$ be a solution to* (P). *For sufficiently large n there exists $u_n : [-r, \mathrm{T}] \to \mathrm{D}_\Lambda(\theta, p)$ a solution of (P_n) and we have $\lim_{n\to\infty} u_n = u$ in $\tilde{\mathrm{X}}_{-r,\mathrm{T}}$.*

3. REGULARITY AND EXISTENCE IN THE LARGE OF THE SOLUTION

The regularity of the mild solution can be studied by using the theory of Da Prato–Grisvard [1]; the following theorem can be proved with the aid of Theorem 4.7 of [1].

THEOREM 4. *Let $\mathcal{F}$ verify the condition:*

(C) *there exist $p' \in [1, +\infty]$ and $\theta' \in]0, 1[$ such that for every $\mathrm{T} > 0$ and $u \in \tilde{\mathrm{X}}_{-r,\mathrm{T}}$, $t \to \mathcal{F}(t, v(\cdot))$ is continuous from $[0, \mathrm{T}]$ to $\mathrm{D}_\Lambda(\theta', p')$.*

Then every mild solution is classical i.e. for each $t > 0$, $u(t) \in D_A$, *u is continuously differentiable from* $]0, T]$ *to* E *and* $u'(t) = Au(t) + \mathscr{F}(t, u(\cdot))$ *holds.*

The asymptotic behaviour of the maximal mild solution can be described by the following theorem, which is not difficult to prove:

THEOREM 5. *If* $u^* : [-r, T^*[\rightarrow D_A(\theta, p)$ *is the maximal solution of* (P) *and* $T^* < +\infty$, *we have* $\sup \{ \| u^*(t) \|_{\theta,p}, t \in [-r, T^*[\} = +\infty$.

We give now some results concerning the existence in the large of the solution.

We need the following Lemma, whose proof is similar to that of Theorem 4.1. of [4]. If $x, y \in E$, let $D_+ \| x \| y$ denote the right derivative of the function $t \rightarrow \| x + ty \|$ at zero.

LEMMA. *Let* $\mathscr{F}$ *verify the condition*:

(F_3) *For every* $T > 0$ *and* $u \in \tilde{X}_{-r,T}$, $D_+ \| u(t) \| \mathscr{F}(t, u(\cdot)) \leq 0$, $t \in [0, T]$.

Then for each solution of (P) *in* $[-r, T]$ *we have* $\| u(t) \| \leq \| u(0) \|$ *for every* $t \in [0, T]$.

Remark. A simple case in which (F_3) is satisfied is when we have $u_1 \cdot H(t, s, x, u_1, u_2, w_1, w_2) \leq 0$ in A (for example $H = -u_1 | w_2 |$).

Now we give a last theorem about the global existence of the solution.

THEOREM 6. *Suppose* (F_3) *hold and let there exist* a, $a_{i,j}$ $(i, j = 1, 2)$, *continuous functions from* $\{(t, s, x), -r = s \leq t < +\infty, x \in \overline{\Omega}\}$ *into* R_+ *such that*

$$(F_4) \quad | H(t, s, x, u_1, u_2, w_1, w_2) | \leq a(t, s, x) | u_1 \cdot u_2 | + \sum_{i,j}^{1,2} a_{i,j}(t, s, x) | u_i | \cdot | w_j |$$

in A. *Then for any* $h \in \tilde{X}_{-r,0}$ *there exists a unique solution to the problem* (P) *in* $[-r, +\infty]$.

In fact we can deduce from the lemma and from (F_4) that $\sup \{ \| u(t) \|_{\theta,p}, t \in [-r, T[\} < +\infty$. The conclusion follows from Theorem 5.

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