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Gravitational field theory for the continuum: second order field equations

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Teorie relativistiche. — *Gravitational field theory for the continuum: second order field equations* (*). Nota di GIANCARLO SPINELLI, presentata (**) dal Socio C. CATTANEO.

RIASSUNTO. — Una teoria del campo gravitazionale generato da un mezzo materiale continuo, può essere anche formulata a partire dallo spazio–tempo pseudo–euclideo nonrinormalizzato, come una teoria di campo nella quale il potenziale gravitazionale è rappresentato da un tensore doppio simmetrico $\psi_{\alpha\beta}$. Avendo adottato, per comodità, una formulazione variazionale, la natura continua della materia gravitante introduce vincoli nuovi rispetto al noto caso della particella puntiforme. La teoria viene costruita in modo iterativo; nella presente Nota vengono dati gli sviluppi dettagliati, di possibile utilità applicativa, sino al secondo ordine.

1. INTRODUCTION

It is well known that nowadays there are many different [1] approaches to the theory of general relativity. One of them, particularly developed in the last fifteen years, treats gravity as a usual field theory starting from flat, unrenormalized space-time [2] (i.e. the space that would appear to an ideal observer using ideal rods and clocks, unaffected by gravity). Such a theory was constructed for interacting point-like particles in an iterative way, requiring consistency to each step of the iteration, and assuming as field source the energy-momentum tensor of the preceding order of approximation. It has been shown that after a renormalization (corresponding to the use of real clocks and rods, affected by gravity) this theory converges [3] to Einstein's theory, even if the most general divergenceless tensor is added [4] to the source.

Two generalizations naturally arise. The first one is to consider particles whose rest mass depends on the gravitational potential. This has been performed in another paper and leads to a theory practically equal to Einstein's theory if the recent stochastic interpretation of quantum electrodynamics [5] is used.

The second generalization is the aim of the present paper. The theory for the gravitational field of an adiabatic continuum is formulated in the same spirit of the flat space-time approach developed so far for point-like particles only. A new kind of problems arises because in the variational formulation one has to take into account the constraints coming from the continuity equation which, in turn, has to be implemented with new terms to each order of approximation.

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Here the problem is limited to find the gravitational field equations. The equations of motion which appear, are only finalized to the deduction of the field equations; hence, in the former equations the stress tensor will be treated as a given field. The problem of the link between stress and strain is not considered. Indeed, the final result to which the present approach converges will be the general relativity equations for a continuum and the problem of the constitutive equations can be tackled at this level. For the bibliography on the latter problem, as well as for its treatment in a complete theory on the motion of an elastic, adiabatic continuum in the general theory of relativity, see the article by Cattaneo [6].

2. Zeroth order approximation : Absence of gravity

Our procedure is an iterative one. In the zeroth order step a neutral continuum in the absence of gravity is considered. In other words we consider the formulation of the dynamics of a continuum in special relativity.

In view of the following steps of the theory it is convenient to give a variational formulation starting from the principle

$$0 = \delta \int \sqrt{-a}\, d^4 x L \,, \tag{1}$$

where (x^α) is a general coordinate system of the pseudo-Euclidean space-time, a the determinant of the matrix $| a_{\alpha\beta} |$, $a_{\alpha\beta}$ the fundamental metric tensor reducing, in orthogonal Cartesian coordinates to $\eta_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$, and L the Lagrangean density.

The equations of motion are obtained by letting vary in (1) the dynamical variables [7] i.e. the coordinates z^α of the matter element. Such z^α are not free but they are subjected to two constraints. The first one is given by

$$\dot{z}^\alpha \dot{z}_\alpha = 1 \,, \tag{2}$$

where $\dot{z}^\alpha = dz^\alpha/ds$, $ds^2 = a_{\alpha\beta}\, dz^\alpha\, dz^\beta$ and we have put the light speed $c = 1$.

A second constraint arises because of the energy balance. Let us consider a proper volume element dV_0 containing a proper mass $\mu_0\, dV_0$ where μ_0 is the proper density of proper mass. When varying z^α, dV_0 is varied. Considering only adiabatic transformations we equate the variation of the energy contained in an infinitesimal proper volume with the work done by the stress tensor $S_{\alpha\beta}$ (which reduces to the classical stress tensor t_{rs} with $r, s = 1, 2, 3$ in a local comoving system). We have ($c = 1$, and semicolons stand for covariant differentiation)

$$\delta_z (\mu_0\, dV_0) = S_{\alpha\beta} (\delta z^\alpha)^{;\beta}\, dV_0 \,, \tag{3}$$

since $(\delta z^\alpha)^{;\beta}$ is the strain relative to the unvaried configuration, i.e. the displacement per unit length owing to which the components of the force $S_{\alpha\beta}$ per unit surface perform a work per unit volume.

When considering real displacements, where $d(dV_0)/ds = dV_0\, \dot{z}^{\beta}{}_{;\beta}$, taking into account that $S_{\alpha\beta}\, \dot{z}^{\alpha} = 0$, one gets

$$(4) \qquad (\mu_0\, \dot{z}_\beta)^{;\beta} + S_{\alpha\beta}{}^{;\beta}\, \dot{z}^{\alpha} = 0\,,$$

which is a power balance and can be considered a continuity equation.

In eq. (1) L is different from zero only where matter is present, hence we can write $\sqrt{-a}\; d^4 x = ds\, dV_0$ and perform the variation as follows

$$(5) \qquad 0 = \delta_z \int ds\, dV_0\, \mu_0\, \bar{L} = \int \delta_z (ds)\, \mu_0\, dV_0\, \bar{L} + \int ds\, \bar{L}\, \delta_z (\mu_0\, dV_0) + \int ds\, \mu_0\, dV_0\, \delta_z (\bar{L})\,,$$

where $\bar{L} = L/\mu_0$.

By (2). (3) and (5) we get [8]

$$(6) \qquad \frac{d}{ds}\left[\mu_0 \frac{\partial \bar{L}}{\partial \dot{z}^{\alpha}}\right] - \mu_0 \frac{\partial \bar{L}}{\partial z^{\alpha}} + \left[\left(\bar{L} - \mu_0 \frac{\partial \bar{L}}{\partial \dot{z}^{\beta}}\, \dot{z}^{\beta}\right) \dot{z}^{\gamma}\, \dot{z}_{\alpha}\right]_{;\gamma} + \mu_0 \frac{\partial \bar{L}}{\partial \dot{z}^{\alpha}}\, \dot{z}^{\gamma}{}_{;\gamma} + (\bar{L}\, S_{\alpha\beta})^{;\beta} = 0\,.$$

In the case of a continuum in the absence of gravity, by the Lagrangean density

$$(7) \qquad L = -\mu_0\,,$$

we obtain the equations of motion

$$(8) \qquad \frac{D}{ds}(\mu_0\, \dot{z}_\alpha) + \mu_0\, \dot{z}_\alpha\, \dot{z}_\beta{}^{;\beta} + S_{\alpha\beta}{}^{;\beta} = 0\,,$$

where D/ds denotes total covariant differentiation.

Another way of obtaining the equations of motion (8) is by equating to zero the divergence of the energy-momentum tensor. When we are dealing with point-like particles the energy-momentum tensor can be obtained [9, 10] by $T_{\alpha\beta} = 2\,(-a)^{-1/2}\, \delta (L \sqrt{-a})/\delta a^{\alpha\beta}$. However, here it is much more convenient to obtain $T_{\alpha\beta}$ in a different way.

Varying the coordinates x^{α} in (1) implies variations for z^{α}, $a^{\alpha\beta}$ and for the fields if present. The corresponding variations induced by z^{α} are zero because of (5) (which implies the equations of motion). What obtained by varying the field potentials (e.g. in the following steps where the gravitational field will be present) is zero because of the field equations. Only the variations induced by $\delta a^{\alpha\beta}$ remain [9, 10].

In the present case $(L = -\mu_0)$ it is

$$(9) \qquad -\delta_a \int \mu_0\, ds\, dV_0 = -\int ds\, \delta_a (\mu_0\, dV_0) - \int \mu_0\, dV_0\, \delta_a (ds)\,,$$

where the subscript a points out that we are varying the fundamental metric tensor $a^{\alpha\beta}$ and $(\mu_0\, dV_0)$ is the mass contained in the proper volume element dV_0. Such a mass changes if work is done on the element. Varying $a^{\alpha\beta}$ we have a deformation tensor given [11] by $\delta a^{\alpha\beta}/2$, hence, from an energy balance

$$\delta_a (\mu_0\, dV_0) = - S_{\alpha\beta} \frac{\delta a^{\alpha\beta}}{2}\, dV_0 . \tag{10}$$

By (9) and (10) and taking into account that $\delta_a (ds) = \frac{1}{2} \dot{z}^\alpha \dot{z}^\beta \delta a_{\alpha\beta}\, ds$, because of the arbitrariness of $\delta a^{\alpha\beta}$ the zeroth step energy-momentum tensor

$$T^{(0)}_{\alpha\beta} = \mu_0 \dot{z}_\alpha \dot{z}_\beta + S_{\alpha\beta} , \tag{11}$$

is obtained. Its divergence equated to zero gives, as required, the equations of motion (8).

3. The theory in the presence of the gravitational field

Let us now take into account that the matter produces, and is subjected to, a gravitational field. Our approach is a field theoretical one in the pseudo-Euclidean " unrenormalized " [2] space-time (i.e. the space that would apper to an ideal observer using ideal rods and clocks, unaffected by gravity). Gravity is represented by a second rank symmetric tensor $\psi_{\alpha\beta}$.

The method by which the theory is constructed is iterative and the first order field equations are immediatly obtained. Because of the equivalence principle, one takes as the source of gravity, to the right hand side (RHS) of such equations, the energy-momentum tensor in the absence of gravity (i.e. the one given by eq. (11)). Denoting by f the gravitational coupling constant, it is [12]:

$$\Box\psi^{\alpha\beta} - \psi_\sigma{}^{(\alpha;\beta)\sigma} + \psi^{;\alpha\beta} + a^{\alpha\beta} (\psi^{\sigma\lambda}{}_{;\sigma\lambda} - \Box\psi) = f (\mu_0 \dot{z}^\alpha \dot{z}^\beta + S^{\alpha\beta}) . \tag{12}$$

The LHS is the usual one [13] coming from the variational formulation and determined by gauge invariance arguments for free fields.

The Dicke framework is here accepted which requires the theory to come from a variational principle [14]. Therefore, the problem is to find an action integral such that varying $\psi_{\alpha\beta}$ in it, and equating to zero the result, should give the field equations (12).

Let us split the Langrangean density as

$$L = L_F + L_M , \tag{13}$$

where L_F is the part due to the fields only, different from zero even where matter is not present, while L_M (relevant to the matter and its interaction with fields) is different from zero only where matter is present. Hence the position

$\sqrt{-a}\, d^4 x = ds\, dV_0$ is possible only for the terms relevant to L_M and we can split

$$I = \int \sqrt{-a}\, L_F\, d^4 x + \int L_M\, ds\, dV_0 . \tag{14}$$

When varying $\psi_{\alpha\beta}$ in (14) we have to take into account another effect. Namely, in the "unrenormalized" [2] description real objects (rods and clocks enclosed) are deformed by the gravitational field. Varying the latter, the matter element is subjected to a deformation tensor given [15] by $f\delta\psi_{\alpha\beta}$ owing to which a work is done by the stresses. Consequently, the proper mass of the element is changed by an amount

$$\delta_\psi (\mu_0\, dV_0) = f S^{\alpha\beta}\, \delta\psi_{\alpha\beta}\, dV_0 . \tag{15}$$

In order to obtain eq. (12) by varying $\psi_{\alpha\beta}$ in (14) and equating to zero the result (i.e. by $\delta_\psi I = 0$), it must be (up to a divergenceless term):

$$L_F = \tfrac{1}{2} \psi_{\mu\nu;\gamma}\, \psi^{\mu\nu;\gamma} - \psi_{\mu\nu;\gamma}\, \psi^{\mu\gamma;\nu} + \psi_{\mu\nu}{}^{;\nu}\, \psi^{;\mu} - \tfrac{1}{2} \psi_{;\mu}\, \psi^{;\mu} , \tag{16}$$

and

$$L_M = \mu_0 (-1 + f\psi_{\rho\lambda}\, \dot{z}^\rho\, \dot{z}^\lambda) . \tag{17}$$

Now, since the action integral is known, the equations of motion could be obtained by varying the dynamical variables z^α analogously to what performed in Sect. 2. The fact is that here we do not a priori know the expression of the continuity equation. The alternative way is to get the equations of motion through the energy-momentum tensor relevant to the present step of approximation. This tensor is obtained by varying the fundamental metric tensor $a_{\alpha\beta}$ in the action integral. The splitting made in eq. (14) is convenient in this case too. As to the variation of $\sqrt{-a}\, L_F$ we have to take into account also the derivatives of $a_{\alpha\beta}$ present in the Christoffel symbols implied by the covariant derivatives of $\psi_{\mu\nu}$ (see Ref. 10). The second term in the RHS of eq. (14), explicitely given by

$$\int \mu_0 (-1 + f\psi_{\rho\lambda}\, \dot{z}^\rho\, \dot{z}^\lambda)\, ds\, dV_0 . \tag{18}$$

contains $a_{\alpha\beta}$ in $ds = (a_{\mu\nu}\, dz^\mu\, dz^\nu)^{1/2}$ and in $\dot{z}^\alpha = dz^\alpha/ds$. Moreover, here too, when varying $\delta_a (\mu_0\, dV_0)$ eq. (10) has to be taken into account. We get eventually the first step energy-momentum tensor

$$\begin{aligned} T^{(1)}_{\alpha\beta} = {} & \psi_{\rho\sigma;\alpha}\, \psi^{\rho\sigma}{}_{;\beta} - 2\, \psi_{\rho\gamma;(\alpha}\, \psi_{\beta)}{}^{\rho;\gamma} + 2\, \psi_{\rho(\alpha}\, \psi_{\beta)\lambda}{}^{;\rho\lambda} + 2\, \psi_{\alpha\rho;\lambda}\, \psi_\beta{}^{\lambda;\rho} - \\ & - \psi_{\rho(\alpha}\, \psi_{;\beta)}{}^{\rho} - 2\, \psi_{\alpha\beta;\gamma\lambda}\, \psi^{\gamma\lambda} + 2\, \psi_{\alpha\gamma;\rho}\, \psi_\beta{}^{\gamma;\rho} - 2\, \psi_{\alpha\beta;\gamma}\, \psi_\mu{}^{\gamma;\mu} - \\ & - \psi_{\alpha\beta}\, \psi_{\sigma\lambda}{}^{;\sigma\lambda} - \psi_{\alpha\beta;\gamma}\, \psi^{;\gamma} + \psi_{;(\alpha}\, \psi_{\beta)\mu}{}^{;\mu} - \psi_{;\alpha}\, \psi_{;\beta} + a_{\alpha\beta} \big(\tfrac{1}{2} \psi_{;\rho}\, \psi^{;\rho} - \\ & - \tfrac{1}{2} \psi_{\rho\sigma;\tau}\, \psi^{\rho\sigma;\tau} + \psi_{\mu\nu;\lambda}\, \psi^{\mu\lambda;\nu} + \psi_{;\mu\rho}\, \psi^{\mu\rho}\big) + \mu_0\, \dot{z}_\alpha\, \dot{z}_\beta + S_{\alpha\beta} - \\ & - \tfrac{1}{2} f\psi_{\alpha\beta} (\mu_0 + S) + f\psi_{\gamma\lambda}\, \mu_0\, \dot{z}^\gamma\, \dot{z}^\lambda\, \dot{z}_\alpha\, \dot{z}_\beta - f\psi_{\gamma\lambda}\, \dot{z}^\gamma\, \dot{z}^\lambda\, S_{\alpha\beta} . \end{aligned} \tag{19}$$

Now that the energy-momentum tensor is known, the first order equations of motion can be obtained by $T^{(1);\beta}_{\alpha\beta} = 0$ substituting in it the field equations and $T^{(1);\beta}_{\alpha\beta}\, \dot{z}^{\alpha} = 0$ which can be considered a continuity equation. We get for the latter

$$(20) \qquad (\mu_0\, \dot{z}^{\gamma})_{;\gamma} = -S_{\alpha\beta}{}^{;\beta}\, \dot{z}^{\alpha} - (2\, f\psi_{\lambda\alpha}\, S_{\beta}{}^{\lambda})^{;\beta}\, \dot{z}^{\alpha} + f S^{\gamma\lambda}\, \psi_{\gamma\lambda;\alpha}\, \dot{z}^{\alpha}\,,$$

and for the equations of motion

$$(21) \qquad [\mu_0\, \dot{z}^{\beta}\,(1 + f\psi_{\gamma\lambda}\, \dot{z}^{\gamma}\, \dot{z}^{\lambda})\, \dot{z}_{\alpha} - 2\, f\mu_0\, \psi_{\lambda\alpha}\, \dot{z}^{\lambda}\, \dot{z}^{\beta}]_{;\beta} + f\mu_0\, \psi_{\gamma\lambda;\alpha}\, \dot{z}^{\gamma}\, \dot{z}^{\lambda} +$$
$$+ [S_{\alpha\beta}\,(1 - f\psi_{\gamma\lambda}\, \dot{z}^{\gamma}\, \dot{z}^{\lambda})]^{;\beta} - (2\, f\psi_{\alpha}{}^{\lambda}\, S_{\beta\lambda})^{;\beta} + f S^{\gamma\lambda}\, \psi_{\gamma\lambda;\alpha} = 0\,.$$

The same equations can be got by varying the dynamical variables in the action integral, that is by (1) in which L is given by eq. (16) plus (17), if one assumes

$$(22) \qquad \delta_z\,(\mu_0\, dV_0)/dV_0 = -S_{\alpha\beta}\,(\delta z^{\alpha})^{;\beta} - (2\, f\psi_{\lambda\alpha}\, S_{\beta}{}^{\lambda})^{;\beta}\, \delta z^{\alpha} + f S^{\gamma\lambda}\, \psi_{\gamma\lambda;\alpha}\, \delta z^{\alpha}$$

which implies (20). This second procedure would be more convenient in view of higher order approximations since it implies the use of Lagrangean densities of the same order as the equations of motion to be deduced. On the contrary the energy momentum tensor is, at each step, one order (in $f\psi$) higher than the equations of motion to be deduced. However it seems very difficult to obtain (22) directly by physical reasoning.

We notice here that, as usual in this kind of flat space-time approach, the first order field equations (12) are not consistent with the equations of motion (21). Indeed the divergence of the LHS of eqs. (12) is zero while the divergence of the RHS equated to zero implies eqs. (8) as if gravity where absent. To get second order consistency the energy-momentum tensor (19) has to be substituted for the source of the eqs. (12), which gives

$$(23) \qquad \Box\psi^{\alpha\beta} - \psi_{\sigma}{}^{(\alpha;\beta)\sigma} + \psi^{;\alpha\beta} + a^{\alpha\beta}(\psi^{\sigma\lambda}{}_{\sigma\lambda} - \Box\psi) = f T^{(1)\alpha\beta}\,.$$

Now the procedure becomes iterative. One finds the Lagrangean density to be put in the action integral such that the second order field equations (23) can be obtained by varying $\psi_{\alpha\beta}$ in (1). Then, varying $a_{\alpha\beta}$ the corresponding energy-momentum tensor is obtained. Its divergence equated to zero, (using the second order field equations), gives the second order equations of motion which, again, are not consistent with eqs. (23) and so on.

Once the iteration is begun, it must be continued to all orders [3]. Indeed, to each order there is inconsistency between the field equations and the equations of motion. The consistency will be reached only when considering the full series. It will be shown in a subsequent paper that even in this case of the continuum the procedure converges to general relativity. Here the second order approximation only has been calculated, which can be useful for practical applications since, apart from black holes, all the known relativistic effects are measurable at maximum with second order precision.

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[11] Directly by the definition of the deformation tensor. See for example L. D. LANDAU and E. M. LIFSHITZ (1959) – *Theory of Elasticity*, (London) Chapt. 1.

[12] Parentheses containing two indices, denote symmetrization, e.g. $\psi_{\alpha(\beta;\gamma)} = \psi_{\alpha\beta;\gamma} + \psi_{\alpha\gamma;\beta}$. The traces of tensor are written by suppresing the repeated indices e.g. $\psi_\sigma{}^\sigma = \psi$. Finally $\Box$ is the d'Alembertian operator i.e. $\Box\psi_{\alpha\beta} = \psi_{\alpha\beta;\lambda}{}^{\lambda}$.

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[15] As shown in Ref. [2] an atom put in the gravitational field, undergoes, in the linear approximation, a deformation given by a tensor $f\psi_{\alpha\beta}$. It is the same deformation to which real rods and clocks (made out of atoms) are subjected, so that a real observer does not measure a pseudo–Euclidean but a Riemannian space–time. Taking into account that the matter is made out of atoms, all the objects are deformed by gravity in the unrenormalized picture. Hence, in such space–time a variation $\delta\psi_{\alpha\beta}$ causes an increase of the deformation tensor equal to $f\delta\psi_{\alpha\beta}$.