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Ludvik Janos

Topological dimension as a first order property

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Topologia. — *Topological dimension as a first order property.* Nota di LUDVIK JANOS, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si studiano alcune proprietà degli spazi separabili di Hilbert.

1. INTRODUCTION

It has been shown recently by C. W. Henson, C. G. Jockusch, L. A. Rubel and G. Takeuti in their paper " First order topology " [1] that topological dimension along with many other important topological properties can be presented as a first order property via the language L_s corresponding to a suitable structure S (X) (as, e.g. the lattice $\mathscr{L}$ (X) of all closed subsets of a space X or the ring C (X) of all bounded continuous functions on X etc.) associated with a topological space X. The purpose of this note is to show that the separable Hilbert space l_2 with its rich linear structure provides another means for expressing dimension, if we restrict our attention to the class C of separable metric spaces. Using the powerful geometrical results of J. H. Roberts [6] we shall show that the interaction between subsets U of l_2 which are homeomorphic to a given space $X \in C$ and the affine subspaces of l_2 determines the dimension of X in terms of sentences of an appropriate first order language L.

For the comfort of the reader we list here all the pertaining facts and concepts which we shall use in the sequel (see [2], [3] or [4]).

DEFINITION 1.1. *Let* (X, d) *be a metric space. A subset* Y *of* X *is called a bisector set in* X *iff there are distinct points* $x_1, x_2 \in X$ *such that* $Y = \{y : d(y, x_1) = d(y, x_2)\}$.

DEFINITION 1.2. *Let* (X, d) *be a metric space. We write* $Y \rhd Z$ *iff* $Z \subset Y \subset X$ *and* Z *is a bisector in* Y *relative to the metric induced on* Y *by* d.

This gives rise to the concept of a *chain* $Y \rhd Y_1 \rhd Y_2 \cdots \rhd Y_n$ in a metric space (X, d). We say that a chain $X = X_0 \rhd X \rhd \cdots \rhd X_{n-1} \rhd X_n$ in (X, d) is a *reduced chain of length n* if $\dim(X_n) \leq 0$ and $\dim(X_{n-1}) > 0$. Here and in the sequel by dim (X) we denote the covering dimension of a space X (see [5]). Thus, the condition $\dim(X_n) \leq 0$ means that the last member X_n in the chain is either empty or zero-dimensional.

By $r(X, d)$ is denoted the maximum length n of reduced chains in (X, d), and for a metrizable space X we define $r(X)$ as the minimum of $r(X, d)$ where the minimum is taken over the set of all metrizations

(*) Nella seduta del 15 giugno 1978.

of X. In case that X is separable, i.e., $X \in C$ there are totally bounded metrics on X and we define $t(X)$ as the minimum of $r(X, d)$ taken over the set of all totally bounded metrizations of X.

THEOREM 1.A (H. Martin). *On the class* C *the function* $t(X)$ *coincides with* dim (X).

For the proof see [4].

LEMMA 1.1. *Assume* $Y \triangleright Y_1 \triangleright \cdots \triangleright Y_n$ *is a chain in a metric space* (X, d). *Then there is a chain* $X \triangleright X_1 \triangleright \cdots \triangleright X_n$ *in* (X, d) *such that* $Y_i = X_i \cap Y$ *for* $i = 1, \cdots, n$.

For the proof, which is easy, see [3] Lemma 2.1.

DEFINITION 1.3. *For* $n = 1, 2, \cdots$ *we denote by* $\mathscr{A}_n$ *the set of all n-dimensional affine subspaces of* l_2, *where by an affine subspace we mean a translate of a linear subspace of* l_2. *By* $\mathscr{A}$ *we denote the untion* $\bigcup_{n=1}^{\infty} \mathscr{A}_n$.

DEFINITION 1.4. *For a space* $X \in C$ *we denote by* $\mathscr{U}(X)$ *the set of all bounded subsets of* l_2 *which are homeomorphic to* X.

THEOREM 1.B (J. H. Roberts). *Assume that* $\dim(X) = n$ *where* $X \in C$ *and* $n \geq 0$. *Then there is* $U \in \mathscr{U}(X)$ *and* $A \in \mathscr{A}_{2n+1}$ *such that*

(1) $U \subset A$ *and*

(2) *for every* $B \in \mathscr{A}_{n+1}$ *the intersection* $U \cap B$ *has dimension* ≤ 0.

Proof. Theorem 1.B is precisely Theorem 1.2 of [6] formulated in terms introduced above. The fact that U can be chosen bounded is apparent from its original proof.

We use the standard terminology of logic, as in [1]. In particular a first order language L is determined by specifying its non-logical symbols, which in our case are unary or binary predicate symbols. The formulas of the language L are built up in the standard way from the predicate symbols, variables, parentheses, propositional connectives $\neg, \wedge, \vee, \rightarrow$, and the quantifiers $(\forall x)$, $(\exists x)$. A sentence is a formula without free variables.

2. INTERACTIONS BETWEEN SUBSETS U OF l_2 AND AFFINE SUBSPACES OF l_2

With a space $X \in C$ we associate the relational structure

$$S(X) = [\mathscr{U}(X); R_0, R_1, \cdots; I_0, I_1, \cdots]$$

having the set $\mathscr{U}(X)$ as its universe on which the unary relations $[R_0, R_1, \cdots$ and $I_0, I_1, \cdots]$ are defined as follows:

For $n \geq 0$ and $U \in \mathscr{U}(X)$ $R_n(U)$ is true iff there is $A \in \mathscr{A}_{2n+1}$ such that $U \subset A$; and $I_n(U)$ is true iff for every $B \in \mathscr{A}_{n+1}$ the intersection $U \cap B$ has dimension ≤ 0.

We denote by L_s the corresponding first order language built on the unary predicate letters R_n^* and I_n^* $(n = 0, 1, \cdots)$ which will always be interpreted as the relations R_n and I_n in $S(X)$ respectively. If ϕ is a sentence of the language L_s and $X \in C$ is a space, we say that "ϕ is true in X" iff $S(X) \models \phi$, i.e., iff the structure $S(X)$ is a model for ϕ.

For $n \geq 0$ we define the sentence ϕ_n of L_s as the sentence

$$(\exists x)\,[R_n^*(x) \wedge I_n^*(x)] .$$

Using ϕ_n we define also sentences ψ_n $(n = 0, 1, \cdots)$ as follows:

$$\psi_0 = \phi_0 \,,\, \psi_1 = \phi_1 \wedge \neg \phi_0 \,, \cdots$$
$$\psi_n = \phi_n \wedge \neg \phi_{n-1} \wedge \cdots \neg \phi_0 .$$

Using these definitions we are now in position to formulate our main result.

THEOREM 2.1. *For* $n \geq 0$ *and* $X \in C$ *the statement* $\dim(X) = n$ *is true if and only if the sentence* ψ_n *is satisfied by* X, *i.e., iff*

$$S(X) \models \psi_n .$$

We give also another alternative of expressing these ideas by letting this time the set $\mathscr{A}$ play the rôle of the universe. For $X \in C$ we introduce the structure $S^1(X) = (\mathscr{A}\,; R^1, \rhd)$ where R^1 is the binary relation on $\mathscr{A}$ defined by: For $A, B \in \mathscr{A}$ $R^1(A, B)$ is true iff there is $U \in \mathscr{U}(X)$ such that:

(*a*) $U \subset A$ and

(*b*) for every B^1 of the same dimension as B the intersection $U \cap B^1$ has dimension ≤ 0.

The symbol $\rhd$ is already known; $A \rhd B$ says that B is a bisector in A relative to the norm-metric in l_2. It is also obvious that for $A, B \in \mathscr{A}$ the statement $A \rhd B$ is equivalent to the statement $B \subset A$ and $\dim(A) - \dim(B) = 1$.

The language for the structure $S^1(X)$ will be denoted by L_{s1}. Thus L_{s1} is based on the two binary predicate symbols R^{1*} and $\rhd^*$ which will always be interpreted by R^1 and $\rhd$ respectively. If we introduce formulas $\alpha_1(x)$, $\alpha_2(x), \cdots$ and $\beta_1(x), \beta_2(x), \cdots$ of one free variable x by

$$\alpha_1(x) = \neg (\exists y)\,[x \rhd^* y] ,$$
$$\alpha_2(x) = (\exists y)\,[x \rhd^* y] , \cdots$$
$$\alpha_n(x) = \exists y_1 \,\exists y_2 \cdots \exists y_{n-1}\,[x \rhd^* y_1 \rhd \cdots \rhd^* y_{n-1}]$$

for $n \geq 2$, and $\beta_n(x) = \alpha_n(x) \wedge \neg \alpha_{n+1}(x)$ for $n = 1, 2, \cdots$, we see easily that the phrase "A has dimension n" can be expressed in L_{s1} as follows:

$A \in \mathscr{A}_n$ iff the formula $\beta_n(x)$ is true in the structure $(\mathscr{A}; \triangleright)$ assuming that x is interpreted by A (the relation R^1 is irrelevant in this case since it is not contained in $\beta_n(x)$).

Denoting by ϕ_n^1 and ψ_n^1 the sentences defined by

$$\phi_n^1 = \exists x \, \exists y \, [R^{1*}(x, y) \wedge \beta_{2n+1}(x) \wedge \beta_{n+1}(y)]$$

$$\psi_0^1 = \phi_0^1 ,$$

$$\psi_1^1 = \phi_1^1 \wedge \neg \phi_0^1 , \cdots$$

$$\psi_n^1 = \phi_n^1 \wedge \neg \phi_{n-1}^1 \wedge \cdots \wedge \neg \phi_0^1 \qquad \text{for} \quad n = 0, 1, \cdots,$$

we can state our second result, expressing the dimension of a space X in the language L_{s^1}.

THEOREM 2.2. *For* $n \geq 0$ *and* $X \in C$ *the statement* $\dim(X) = n$ *is true if and only if the sentence* ψ_n^1 *is satisfied by* X, *i.e., iff*

$$S^1(X) \models \psi_n^1 .$$

3. PROOF OF THE THEOREMS

Proof of Theorem 2.1. For $X \in C$ and $n \geq 0$ assume first that $\dim(X) = n$. Theorem 1.B implies that there is $U \in \mathscr{U}(X)$ and $A \in \mathscr{A}_{2n+1}$ such that for every $B \in \mathscr{A}_{n+1}$ the set $U \cap B$ has dimension ≤ 0. This means that the sentence ϕ_n is satisfied by X. We have to show that if $n > 0$ then ϕ_k is not satisfied for $k < 0$. Assume the contrary. Then there exists $U \in \mathscr{U}(X)$ such that $R_k(U)$ and $I_k(U)$ which means that there exists $A \in \mathscr{A}_{2k+1}$ such that $U \subset A$ and $U \cap B$ has dimension ≤ 0 for every $B \in \mathscr{A}_{k+1}$. Denoting by d the metric on U induced on U by the norm-metric of l_2 we observe that d is a totally bounded metric since U is a bounded subset of a finite-dimensional affine space A. This fact and Theorem 1.A implies that $r(U, d) \geq t(X) = n$. Now consider a reduced bisector chain in (U, d) of length $r = r(U, d) : U \triangleright U_1 \triangleright \cdots \triangleright U_{r-1} \triangleright U_r$. Lemma 1.1 implies that there is a chain in $A : A \triangleright A_1 \triangleright \cdots \triangleright A_r$ for which $U_i = U \cap A_i$ for $i = 1, 2, \cdots, r$. In particular we have $U_{r-1} = U \cap A_{r-1}$. From the definition of the reduced bisector chain we know that U_{r-1} has a positive dimension which implies that A_{r-1} must have dimension greater than $k + 1$ since we assume that $\dim(U \cap B) \leq 0$ for every $B \in \mathscr{A}_{k+1}$. The dimension of A_{r-1} is precisely $2k + 1 - (r - 1) = 2k + 2 - r$. Thus we obtain the inequality $2k + 2 - r > k + 1$ or $k > r$, which yields the desired contradiction since $r \geq n$.

Thus, so far we proved that $\dim(X) = n$ implies that

$$(*) \qquad S(X) \models \phi_n \quad \text{and not} \quad S(X) \models \phi_k \quad \text{for} \quad k < n$$

in case that $n > 0$. This means precisely that $S(X) \models \psi_n$.

Now assume conversely that ψ_n is true in X and set $\dim(X) = m$, where the possibility that m may be infinite is not excluded. Thus, we assume that ϕ_n is true but ϕ_k is not true for $k < n$ in case that $n > 0$. From the fact that ϕ_n is true follows that there exist $U \in \mathcal{U}(X)$ and $A \in \mathcal{A}_{2n+1}$ such that $U \subset A$ which implies that $m = \dim(X) = \dim(U) \leq 2n + 1$, thus, m is finite. The first relation in (*) applied to this situation yields that ϕ_m is true and the second implies that ϕ_k is not true for any $k < m$. Since ϕ_n is true this implies that $m \leq n$ and since we assume that ϕ_k is false for any $k < n$ we conclude that $m = n$ which completes our proof.

In order to prove easily Theorem 2.2 we observe that the sentence ϕ_n has precisely the same meaning in the structure S (X) as the sentence ϕ_n^1 in the structure $S^1(X)$.

LEMMA 3.1. *For* $n \geq 0$ *and* $X \in C$ *we have* $S(X) \models \phi_n$ *iff* $S^1(X) = \phi_n^1$.

Proof. Assume that ϕ_n is true in X. Thus, there is $U \in \mathcal{U}(X)$ such that $R_n(U)$ and $I_n(U)$ implying that there exist $A \in \mathcal{A}_{2n+1}$ such that $U \subset A$ and $\dim(U \cap B) \leq 0$ for every $B \in \mathcal{A}_{n+1}$. Consulting the definition of R_1 in the structure $S^1(X)$ we see that the pair A, B satisfies R^1 and since A and B have the dimensions $2n + 1$ and $n + 1$ respectively we see that the sentence $\phi_n^1 = \exists x \, \exists y \, (R^{1*}(x, y) \wedge \beta_{2n+1}(x) \wedge \beta_{n+1}(x))$ is satisfied in $S^1(X)$. Observing that this argument is reversible we conclude the proof of our assertion.

The proof of Theorem 2.2 now follows from Theorem 2.1 and Lemma 3.1 since the sentence ψ_n^1 is built up from the sentences $\phi_0^1, \phi_1^1, \cdots, \phi_n^1$ in exactly the same way as the sentence ψ_n from the sentences $\phi_0, \phi_1, \cdots, \phi_n$.

CONCLUDING REMARK

With each space $X \in C$ we have associated the set $\mathcal{U}(X)$ and observed how the elements of $\mathcal{U}(X)$ interact with the elements of the fixed set $\mathcal{A}$. As a result of this observation we have obtained the desired information about the property concerned, i.e., about the dimension of X.

A natural question arises whether this procedure can be suitably generalized as to characterize this way other topological properties as well. Assume that P is a property under consideration and suppose that a set $\mathcal{A}_p$ has been chosen chose elements act as " test spaces " for the property P. Assume further that with each space X of some class C^1 we associate a well defined set $\mathcal{U}_p(X)$ whose elements represent the space X in an appropriate way and whose interaction with the test spaces will be considered. The desideratum is to determine whether or not the space X has the property P in terms of sentences describing this interaction.

EXAMPLE. Let $C^1 = C$ and P be compactness. Choosing $\mathcal{A}_p = \{H\}$ where H is the Hilbert cube we assign to each $X \in C$ the set $\mathcal{U}_p(X)$ defined as $\{U : U \subset H$ and U is homeomorphic to $X\}$. The sentence " there is $U \in \mathcal{U}_p(X)$ which is closed in H " expresses compactness of X.

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