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Asymptotic behaviour of perturbed difference equations

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Equazioni funzionali. — *Asymptotic behaviour of perturbed difference equations.* Nota I di PAVEL TALPALARU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Questa Nota riguarda alcuni problemi qualitativi sulle equazioni alle differenze e in particolare questioni concernenti l'equivalenza asintotica di queste equazioni.

1. INTRODUCTION

The theory of difference equations is in a process of continuous development and it is signifiant for its various applications in numerical analysis, physics, control theory and optimization. In recent years, considerable attention has been paid to the development of the qualitative theory for difference equations.

In this paper, we shall give some general results on the asymptotic relationship between the solutions of a linear difference equation and its perturbed nonlinear equation.

The relationship between the asymptotic behaviour of a homogeneous differential equation and a nonhomogeneous perturbation of that differential equation has been widely investigated. The purpose of this paper is to develop a part of those problems for some classes of difference equations. The problems considered in this article are in the general spirit of the investigations of the Author [5], M. Basti and B. S. Lalli [1] and T. G. Hallam [2], [3].

2. NOTATIONS AND DEFINITIONS

Denote by $\mathrm{N}(n_0) = \{n_0, n_0 + 1, \cdots\}$, where n_0 is a natural number or zero; R^k the k-dimensional real euclidean space with norm $|x| = \sum_{i=1}^{k} |x_i|$, $x = (x_1, x_2, \cdots, x_k)$; M^k the space of all $k \times k$ matrices $\mathrm{A} = (a_{ij})$ with norm $|\mathrm{A}| = \max_j \sum_{i=1}^{k} |a_{ij}|$. The identity matrix is denoted by E. We denote by $\Phi = \Phi(\mathrm{N}, \mathrm{R}^k)$ the space of all functions from $\mathrm{N}(n_0)$ into R^k, that is, for each $n \in \mathrm{N}(n_0)$ the value of x at n is $x(n) \in \mathrm{R}^k$. The topology of Φ is the topology of uniform convergence on every set $\mathrm{N}_m(n_0) = \{n_0, n_0 + 1, \cdots \cdots, n_0 + m\}$, $m = 0, 1, \cdots$ that is, $x_i \to x$ as $i \to \infty$ in Φ if and only if $\lim_{i \to \infty} |x_i(n) - x(n)| = 0$ uniformly on every set $\mathrm{N}_m(n_0)$, $m = 0, 1, \cdots$ Note also that Φ is a locally convex space [8, pp. 24-26] with the topology defined by

(*) Nella seduta del 13 maggio 1978.

the following family of seminorms $|x(n)|_m = \sup\{|x(n)|; n \in N_m(n_0), m = 0, 1, \cdots\}$. We let $\Phi_1 = \Phi_1(N, R^k)$ be the Banach space in Φ of all bounded functions from $N(n_0)$ to R^k. The norm in Φ_1 is defined by $|x|_{\Phi_1} = |x(n)|_{\Phi_1} = \sup\{|x(n)|; n \in N(n_0)\}$.

The spaces Φ and Φ_1 were considered by D. Petrovanu [4] in the study of discrete Hammerstein equations and by C. P. Tsokos and W. J. Padgett [7, Ch. V] in the study of random discrete Fredholm and Volterra equations.

We will be interested in establishing asymptotic relationship, between the solutions of unperturbed equation

$$x(n+1) = A(n)x(n) \tag{2.1}$$

and its perturbed nonlinear equation

$$y(n+1) = A(n)y(n) + f(n, y(n)) + g(n, y(n)), Ty(n)) \tag{2.2}$$

where x, y are k-dimensional vectors, $A: N(n_0) \to M^k$ is such that $A(n)$ is nonsingular for all $n \in N(n_0)$, $f: N(n_0) \times D \to R^k$ is, for any $n \in N(n_0)$ continuous as a function of $y \in D$ (D — a region in R^k), $g: N(n_0) \times D \times D \to R^k$ is, for any $n \in N(n_0)$, continuous in the last two arguments, and T is a continuous operator from $\Phi(N, D)$ into $\Phi(N, D)$.

Note that if $X(n)$ is the fundamental matrix of (2.1), then it is the unique solution of the following matrix difference equation

$$X(n+1) = A(n)X(n), \qquad \text{with} \quad X(n_0) = E,$$

and also that $X(n) = A(n-1)A(n-2)\cdots A(n_0)E$, from which, since $A(n)$ is nonsingular, follows that $X^{-1}(n)$ exists for any $n \in N(n_0)$. With respect to the operator T we can impose on it various meanings.

In the following we will be concerned with the study of the asymptotic equivalence of the equations (2.1) and (2.2). In this paper we consider the notion of asymptotic equivalence given by

DEFINITION 2.1. Let $B(n)$ and $C(n)$ be $k \times k$ matrices and $\alpha(n)$ a positive function. We say that the equations (2.1) and (2.2) are asymptotically equivalent if, corresponding to each solution $x = x(n)$ of (2.1), there exists a solution $y = y(n)$ of (2.2) with the property

$$|B(n)y(n) - C(n)x(n)| = o(\alpha(n)), \qquad \text{as} \quad n \to \infty, \tag{2.3}$$

and conversely.

Since our results are formulated in terms of arbitrary matrices $B(n)$, $C(n)$ and an arbitrary function $\alpha(n)$, they offer a greater versatility in obtaining various asymptotic properties for specific classes of discrete equations.

3. A PRELIMINARY RESULT

To establish our main result on asymptotic equivalence, we need to give a lemma which will be used in the following section. In Lemma 3.1 $X(n)$, $B(n)$, $C(n)$ are nonsingular $k \times k$ matrices for $n \in N(n_0)$; P a projection and q satisfies the inequality $1 \leq q < \infty$.

LEMMA 3.1. *Let there exist a constant* $K > 0$ *such that*

$$\sum_{s=n_0}^{n} |B(n)X(n)PX^{-1}(s)C(s)|^q \leq K^q, \qquad n \in N(n_0); \tag{3.1}$$

and suppose that

$$\sum_{s=n_0}^{\infty} |C^{-1}(s)B^{-1}(s)|^{-q} = \infty. \tag{3.2}$$

Then,

$$\lim_{n\to\infty} |B(n)X(n)P| = 0 \tag{3.3}$$

Proof. For any $n \in N(n_0)$ we have $|C^{-1}(n)X(n)P| > 0$ and therefore we may define $h(n) = |C^{-1}(n)X(n)P|^{-q}$ and $g(n) = \sum_{s=n_0}^{n} h(s)$. Then, from the identity

$$C^{-1}(n)X(n)Pg(n) = \sum_{s=n_0}^{n} C^{-1}(n)B^{-1}(n)B(n)\cdot$$
$$\cdot X(n)PX^{-1}(s)C(s)C^{-1}(s)X(s)Ph(s),$$

it follows, by using Hölder's inequality that

$$|C^{-1}(n)X(n)P|g(n) = |C^{-1}(n)X(n)P| \sum_{s=n_0}^{n} h(s) =$$
$$= |C^{-1}(n)B^{-1}(n)| \left[\sum_{s=n_0}^{n} |B(n)X(n)PX^{-1}(s)C(s)|^q\right]^{1/q} \cdot$$
$$\cdot \left[\sum_{s=n_0}^{n} |C^{-1}(s)X(s)Ph(s)|^p\right]^{1/p},$$

where $p^{-1} + q^{-1} = 1$. Since

$$|C^{-1}(s)X(s)Ph(s)| \leq |C^{-1}(s)X(s)P|\,h(s) = |C^{-1}(s)X(s)P|^{1-q}$$

we have

$$|C^{-1}(s)X(s)Ph(s)|^p \leq |C^{-1}(s)X(s)P|^{p(1-q)} = |C^{-1}(s)X(s)P|^{-q} = h(s).$$

From here and (3.1) it follows

$$|C^{-1}(n)X(n)P| \left[\sum_{s=n_0}^{n} h(s)\right]^{1/q} \leq K\,|C^{-1}(n)B^{-1}(n)|, \qquad n \in N(n_0). \tag{3.4}$$

From the identity

$$B(n)\,X(n)\,Pg(n) = \sum_{s=n_0}^{n} B(n)\,X(n)\,PX^{-1}(s)\,C(s)\,C^{-1}(s)\,X(s)\,Ph(s)\,,$$

in an analogous manner, one can show that

$$(3.5) \qquad |\,B(n)\,X(n)\,P\,| \leq K\left[\sum_{s=n_0}^{n} h(s)\right]^{-1/q}, \qquad n \in N(n_0)\,.$$

The conclusion of the Lemma will follow provided it is established that $\sum_{s=n_0}^{\infty} h(s) = \infty$. To prove this, we observe that from (3.4) we have

$$|\,C^{-1}(n)\,X(n)\,P\,|\,[g(n)]^{1/q} \leq K\,|\,C^{-1}(n)\,B^{-1}(n)\,|$$

and therefore $h(n)\,[g(n)]^{-1} \geq K^{-q}\,|\,C^{-1}(n)\,B^{-1}(n)\,|^{-q}$. Since $h(n) = g(n) - g(n-1)$ it follows that $g(n) - g(n-1) \geq K^{-q}\,|\,C^{-1}(n)\,B^{-1}(n)\,|^{-q}\,g(n)$, from where,

$$(3.6) \qquad g(n)\,[1 - K^{-q}\,|\,C^{-1}(n)\,B^{-1}(n)\,|^{-q}] \geq g(n-1)\,, \qquad n \in N(n_0 - 1)\,.$$

If we use the well-known inequality $1 - u \leq \exp(-u)$, from (3.6) one obtains

$$g(n_0) \leq g(n_0+1)\exp[-K^{-q}\,|\,C^{-1}(n_0+1)\,B^{-1}(n_0+1)\,|^{-q}]\,,$$

$$g(n_0+1) \leq g(n_0+2)\exp[-K^{-q}\,|\,C^{-1}(n_0+2)\,B^{-1}(n_0+2)\,|^{-q}]\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$g(n-1) \leq g(n)\exp[-K^{-q}\,|\,C^{-1}(n)\,B^{-1}(n)\,|^{-q}]\,,$$

from where, $g(n) \geq g(n_0)\exp\left[K^{-q}\sum_{s=n_0+1}^{n}|\,C^{-1}(s)\,B^{-1}(s)\,|^{-q}\right]$.

This inequality implies that $\lim_{n\to\infty} g(n) = \sum_{s=n_0}^{n} h(s) = \infty$ which yields the desired conclusion.

Remark. When $B = C = E$ the condition (3.1) becomes

$$(3.1') \qquad \sum_{s=n_0}^{n} |\,X(s)\,PX^{-1}(s)\,|^{-q} \leq K^{q}\,,$$

(3.2) is satisfied and (3.3) becomes

$$(3.3') \qquad \lim_{n\to\infty} |\,X(n)\,P\,| = 0 \qquad \text{(Lemma, [6])}\,.$$

4. Asymptotic equivalence of (2.1) and (2.2)

In this section we shall show that, under some conditions for a given solution $x = x(n)$ of (2.1) there exists at least a solution $y = y(n)$ of (2.2) such that (2.4) holds with $\alpha(n) \equiv 1$ and conversely.

THEOREM 4.1. *Suppose that the following conditions are satisfied*:

a) *there exist two nonsingular* $k \times k$ *matrices* $B(n)$ *and* $C(n)$ *defined for* $n \in N(n_0)$ *such that*

$$|B(n)| \leq M, \; |B^{-1}(n)| \leq M, \; n \in N(n_0), \; |B(n) - C(n)| \to 0$$

as $\quad n \to \infty, \; \sum_{s=n_0}^{\infty} |C^{-1}(s)|^{-q} = \infty, \; q > 1$ (M *is a positive constant*);

b) *there exist two supplementary projections* P_1 *and* P_2 *and a positive constant* K *such that if* $X(n)$ *is a fundamental matrix of the equation* (2.1) *then*

$$\sum_{s=n_0}^{n-1} |B(n) X(n) P_1 X^{-1}(s+1) C(s+1)|^q +$$

$$+ \sum_{s=n}^{\infty} |B(n) X(n) P_2 X^{-1}(s+1) C(s+1)|^q \leq K^q, \qquad n \in N(n_0);$$

c) *there exists a non-negative function* $\omega_1(n, u)$ *defined on* $N(n_0) \times R_+$ *non-decreasing in* u *and such that* $\omega_1(n, a) \in l_p$ *for each* $a \in R_+$ *and*

$$|C^{-1}(n+1) f(n, y(n))| \leq \omega_1(n, |y(n)|), \; n \in N(n_0), \; |y| < \infty;$$

d) *there exists a function* $\omega_2(n, u, v)$ *defined on* $N(n_0) \times R_+ \times R_+$, *non-decreasing in* u *and* v *and such that* $\omega_2(n, a, b) \in l_p$ *for each* $a, b \in R_+$ *and furthemore*

$$|C^{-1}(n+1) g(n, y(n), Ty(n))| \leq \omega_2(n, |y(n)|,$$
$$|Ty(n)|), \; n \in N(n_0), \; |y| < \infty;$$

e) $|Ty| \leq L|y| \quad$ *for* $\quad y \in D$.

Then, corresponding to each bounded solution $x = x(n) \in \Phi_1$ *of* (2.1), *there exists a solution* $y = y(n) \in \Phi_1$ *of* (2.2) *such that*

$$\lim_{n \to \infty} |B(n) y(n) - C(n) x(n)| = 0. \tag{4.1}$$

Conversely, to each solution $y = y(n) \in \Phi_1$ *of* (2.2) *there exists a solution* $x = x(n) \in \Phi_1$ *of* (2.1) *such that* (4.1) *holds.*

Proof. Let $x = x(n)$ be a solution of (2.1) such that $|x| < \rho/3$, $\rho > 0$ for $n \in N(n_0)$ and define the ball $B_\rho = \{u; |u|_{\Phi_1} \leq \rho\}$. For $y \in B_\rho$ we define the operator

$$\tau y(n) = x(n) + \sum_{s=n_0}^{n-1} X(n) P_1 X^{-1}(s+1) [f(s, y(s) + g(s, y(s), Ty(s))] -$$

$$- \sum_{s=n}^{\infty} X(n) P_2 X^{-1}(s+1) [f(s, y(s)) + g(s, y(s), Ty(s))],$$

for $\quad n \in N(n_0 + 1)$.

If we set $\varphi_i(n, s) = B(n) X(n) P_i X^{-1}(s+1) C(s+1)$, $i = 1, 2$, then,

$$
\begin{aligned}
(4.2) \qquad |\tau y(n)| \leq \rho/3 &+ \sum_{s=n_0}^{n-1} |B^{-1}(n)| |\varphi_1(n,s)| |C^{-1}(s+1) f(s, y(s))| + \\
&+ \sum_{s=n_0}^{n-1} |B^{-1}(n)| |\varphi_1(n,s)| |C^{-1}(s+1) g(s, y(s), Ty(s))| + \\
&+ \sum_{s=n}^{\infty} |B^{-1}(n)| |\varphi_2(n,s)| |C^{-1}(s+1) f(s, y(s))| + \\
&+ \sum_{s=n}^{\infty} |B^{-1}(n)| |\varphi_2(n,s)| |C^{-1}(s+1) g(s, y(s), Ty(s))| \leq \\
&\leq \rho/3 + MK \left[\sum_{s=n_0}^{n-1} \omega_1^p(s, \rho)\right]^{1/p} + \\
&+ MK \left[\sum_{s=n_0}^{n} \omega_2^p(s, \rho, L\rho)\right]^{1/p} + MK \left[\sum_{s=n}^{\infty} \omega_1^p(s, \rho)\right]^{1/p} + \\
&+ MK \left[\sum_{s=n}^{\infty} \omega_2^p(s, \rho, L\rho)\right]^{1/p}.
\end{aligned}
$$

By virtue of the properties of ω_1 and ω_2 (c and d) we may choose n_0 so large that

$$\sum_{s=n_0}^{\infty} \omega_1^p(s, \rho) \leq [\rho/6KM]^p \qquad \text{and} \qquad \sum_{s=n_0}^{\infty} \omega_2^p(s, \rho, L\rho) \leq [\rho/6KM]^p$$

and therefore from (4.2) it follows $|y(n)|_{\Phi_1} \leq \rho$, which shows that $\tau(B_\rho) \subset B_\rho$. To establish that the mapping is continuous, fixe $\varepsilon > 0$, and select $n_1 \in N(n_0 + 1)$ such that

$$(4.3) \qquad \left[\sum_{s=n_1}^{\infty} \omega_1^p(s, \rho)\right]^{1/p} + \left[\sum_{s=n_1}^{\infty} \omega_2^p(s, \rho, L\rho)\right]^{1/p} < \varepsilon/2\,KM.$$

Suppose $\{y_i(n)\}_{i=1}^{\infty}$, $y_i \in B_\rho$, and $y_i(n) \to y(n) \in B_\rho$ on every set $N_m(n_0)$, $m = 0, 1, 2, \cdots$ Then, for $n \in N_m(n_0)$ we have

$$
\begin{aligned}
|\tau y_i(n) - \tau y(n)| &\leq \left[\sum_{s=n_0}^{n_1-1} |\varphi_1(n,s)|^q\right]^{1/q} \left\{\left[\sum_{s=n_0}^{n_1-1} [|B^{-1}(n)| |C^{-1}(s+1) \cdot \right.\right. \\
&\cdot [f(s, y_i(s)) - f(s, y(s))]|]^p\Big]^{1/p} + \left[\sum_{s=n_0}^{n_1-1} [|B^{-1}(n)| |C^{-1}(s+1) \cdot \right. \\
&\left.\left. \cdot [g(s, y_i(s), Ty_i(s)) - g(s, y(s), Ty(s))]|]^p\right]^{1/p}\right\} + \\
&+ \left[\sum_{s=n_1}^{\infty} |\varphi_2(n,s)|^q\right]^{1/q} \left\{\left[\sum_{s=n_1}^{\infty} [|B^{-1}(n)| |C^{-1}(s+1) [f(s, y_i(s)) - \right.\right. \\
&- f(s, y(s))]|]^p\Big]^{1/p} + \left[\sum_{s=n_1}^{\infty} [|B^{-1}(n)| |C^{-1}(s+1) [g(s, y_i(s), Ty_i(s)) - \right.
\end{aligned}
$$

$$-g(s, y(s), Ty(s))] |^p\Big]^{1/p}\Big\} \leq MK[G_1 + G_2] +$$

$$+ 2\,KM\left[\left(\sum_{s=n_1}^{\infty}\omega_1^p(s, \rho)\right)^{1/p} + \left(\sum_{s=n_1}^{\infty}\omega_2^p(s, \rho, L\rho)\right)^{1/p}\right] <$$

$$< MK[G_1 + G_2] + \varepsilon,$$

if we choose, according to (4.3) n_1 sufficiently large.

By the uniform convergence on $N_m(n_0)$, $m = 0, 1, \cdots, n_1 - 1$ of $\{y_i(n)\}$ it follows that $f(n, y_i(n)) \to f(n, y(n))$ and $g(n, y_i(n), Ty_i(n)) \to g(n, y(n), Ty(n))$, uniformly on this set, and

$$G_1 + G_2 \leq \left[\sum_{s=n_0}^{n_1-1}(|C^{-1}(s+1)[f(s, y_i(s)) - f(s, y(s))]|)^p\right]^{1/p} +$$

$$+ \left[\sum_{s=n_0}^{n_1-1}(|C^{-1}(s+1)[g(s, y_i(s), Ty_i(s)) - g(s, y(s), Ty(s))]|)^p\right]^{1/p}$$

$$\text{as} \quad i \to \infty.$$

Hence $\lim\limits_{n\to\infty} | \tau y_i - \tau y |_{\Phi_1} = 0$, that is, τ is continuous.

The functions in the image space $\tau(B_\rho)$ are uniformly bounded for each n since $\tau(B_\rho) \subset B_\rho$. The equicontinuity of the family $\tau(B_\rho)$ follows because the functions in $\tau(B_\rho)$ are defined for a discrete variable n.

By Schauder's fixed point theorem we conclude that the mapping τ has a fixed point in B_ρ which is a solution of (2.2). To verify that (4.1) holds, observe that we have

$$(4.4) \qquad | By(n) - Cx(n) | \leq | B(n) - C(n) | \, | x(n) | + H_1 + H_2,$$

where

$$H_1 = \sum_{s=n_0}^{n-1} | B(n) X(n) P_1 X^{-1}(s+1)[f(s, y(s)) + g(s, y(s), Ty(s))] |,$$

$$H_2 = \sum_{s=n}^{\infty} | B(n) X(n) P_2 X^{-1}(s+1)[f(s, y(s)) + g(s, y(s), Ty(s))] |.$$

Using *c*), *d*) and Hölder's inequality we get

$$H_2 \leq K\left[\sum_{s=n}^{\infty}\omega_1^p(s, \rho)\right]^{1/p} + K\left[\sum_{s=n}^{\infty}\omega_2^p(s, \rho, L\rho)\right]^{1/p} < \varepsilon/2,$$

for $n \in N(n_1)$, where n_1 is sufficiently large.

If we take $n_2 \in N(n_1)$ such that

$$\sum_{s=n_2}^{\infty}\omega_1^p(s, \rho) < [\varepsilon/6K]^p, \ \sum_{s=n_2}^{\infty}\omega_2^p(s, \rho, L\rho) <$$

$$< [\varepsilon/6K]^p, \ | B(n) X(n) P_1 | < \varepsilon/6\left[\sum_{s=n_0}^{n_2-1}(| X^{-1}(s+1) f(s, y(s)) | +\right.$$

$$\left. + | X^{-1}(s+1) g(s, y(s), Ty(s)) |)\right],$$

then using our Lemma 3.1 we have

$$
\begin{aligned}
H_1 = & \sum_{s=n_0}^{n_2-1} | B(n) X(n) P_1 X^{-1}(s+1) f(s, y(s)) | + \\
& + \sum_{s=n_2}^{n-1} | B(n) X(n) X^{-1}(s+1) f(s, y(s)) | + \\
& + \sum_{s=n_0}^{n_2-1} | B(n) X(n) P_1 X^{-1}(s+1) g(s, y(s), Ty(s)) | + \\
& + \sum_{s=n_2}^{n-1} | B(n) X(n) P_1 X^{-1}(s+1) g(s, y(s), Ty(s)) | \leq \\
& \leq K \left[\sum_{s=n_2}^{n-1} \omega_1^p(s, \rho) \right]^{1/p} + K \left[\sum_{s=n_2}^{n-1} \omega_2^p(s, \rho, L\rho) \right]^{1/p} + \\
& + \sum_{s=n_0}^{n_2-1} | B(n) X(n) P_1 | \, | X^{-1}(s+1) f(s, y(s)) | + \\
& + \sum_{s=n_0}^{n_2-1} | B(n) X(n) P_1 | \, | X^{-1}(s+1) g(s, y(s), Ty(s)) | + \\
& + K \left[\sum_{s=n_0}^{\infty} \omega_1^p(s, \rho) \right]^{1/p} + K \left[\sum_{s=n_2}^{\infty} \omega_2^p(s, \rho, L\rho) \right]^{1/p} + \\
& + | B(n) X(n) P_1 | \left[\sum_{s=n_0}^{n_2-1} (| X^{-1}(s+1) f(s, y(s)) | + \right. \\
& \left. + | X^{-1}(s+1) g(s, y(s), Ty(s)) |) \right] < \varepsilon/2 , \qquad \text{for } n \in N(n_2) .
\end{aligned}
$$

Consequently for sufficiently large n_2 we have $H_1 + H_2 < \varepsilon$ and $| B(n) - C(n) | < \varepsilon/\rho$ if $n \in N(n_2)$, and from (4.4) we obtain $| B(n) y(n) - C(n) x(n) | \to 0$ as $n \to \infty$.

The last statement of the theorem follows immediately. Let $y = y(n)$ be a solution of (2.2). Define

$$
\begin{aligned}
(4.5) \quad x(n) = y(n) - & \sum_{s=n_0}^{n-1} X(n) P_1 X^{-1}(s+1) [f(s, y(s)) + g(s, y(s), Ty(s))] + \\
& + \sum_{s=n}^{\infty} X(n) P_2 X^{-1}(s+1) [f(s, y(s)) + g(s, y(s), Ty(s))] .
\end{aligned}
$$

By the same arguments, from (4.5) it follows that $x = x(n)$ is a solution of (2.1) which belongs to Φ_1 and satisfies (4.1).

Thus the proof of Theorem 4.1 is accomplished.

Remarks 4.1. If we assume that $C(n)$ is such that $| C^{-1}(n) | \leq M_1$, then *c*) and *d*) hold if $| f(n, y(n)) | \leq \bar{\omega}_1(n, | y(n) |)$ and $| g(n, y(n), Ty(n)) | \leq \bar{\omega}_2(n, | y(n) |, | Ty(n) |)$ for $n \in N(n_0)$, $| y | < \infty$, where $\bar{\omega}_1$ and $\bar{\omega}_2$ are of the same type as in Theorem 4.1.

4.2. Conditions which assure e) in the particular cases when $Ty(n) = \sum_{s=n_0}^{\infty} K(n, s)\, h(y(s))$ and $Ty(n) = \sum_{s=n_0}^{n} K(n, s)\, h(y(n))$ are given in [7, Ch. V].

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