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Spectral properties of bounded semi-Fredholm operators

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Analisi funzionale. — *Spectral properties of bounded semi-Fredholm operators* (*). Nota di LUCIANO BARBANTI, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si prova che, se una componente aperta, connessa, massimale, Δ, del dominio di semi-Fredholm di un operatore T (semi-Fredholm, limitato), non contiene autovalori, allora non esistono autovalori in un opportuno intorno aperto, I, di Δ, con $\overline{\Delta} \subset I$. Il risultato viene applicato ad un sistema di equazioni differenziali del tipo neutro, a coefficienti periodici.

I. – INTRODUCTION

The decomposition of the space by means of eigenvalues of finite type is a crucial technique in the theory of Neutral Functional Differential Equations (NFDE). A good decomposition of the initial data space enables us frequently to consider the NFDE, restricted to the finite-dimensional space of the decomposition, as an ODE. This can be seen in the theory of the linear autonomous NFDE. Here, $R_\lambda(A)$, the resolvent of A, is a compact operator (A being the infinitesimal generator of $[T(t)]_{t\geq 0}$, the semigroup of solution-operators of the equation) and consequently, the spectrum of A reduces to eigenvalues which are isolated in $\rho(A)$ (the resolvent set associated with A), and is of finite type.

For the nonautonomous NFDE, instead, the problem of decomposing the initial data space, is much more complicated. However, it is reasonable to try the localization in the resolvent, of isolated eigenvalues of finite type, for classes of operators in which we can classify $T(t, s)$, the solution-operators of a nonautonomous NFDE.

In the following we state, in II, some spectral properties for bounded semi-Fredholm operators, and in III, the results of II are applied to obtain a particular representation for the solutions of a linear periodic NFDE.

II. – SEMI-FREDHOLM OPERATORS

In the following, X, Y denote Banach spaces, C (X, Y) the spaces of closed linear operators from X into Y, and B (X, Y) the normed space of bounded linear operators from X into Y. As usual, C (X, X) and B (X, X), are denoted by C (X) and B (X), respectively.

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DEFINITION 1. *Let* $T \in C(X, Y)$. *The nullity of* T (nul T) *is the dimension of* Ker T. *The deficiency of* T (def T) *is the dimension of* $Y/T(X)$.

DEFINITION 2. *Let* $T \in C(X, Y)$. T *is a semi-Fredholm operator* (*or, an operator of semi-Fredholm type*), *if* T (X) *is closed in* Y, *and* nul $T < \infty$ *or* def $T \leq \infty$.

The class of the bounded semi-Fredholm operators from X into Y forms a linear space, which will be denoted by SF (X, Y), SF (X) being the space SF (X, X).

DEFINITION 3. *Let* $T \in SF(X)$. *The semi-Fredholm domain of* T (*denoted by* Δ_T) *is the set*:

$$\Delta_T = \{\lambda \in \mathbf{C} : (T - \lambda I) \in SF(X)\}.$$

It's well known, (Kato [1], p. 243), that Δ_T is a nonempty open subset of C, for every $T \in SF(X)$. Then, $\Delta_T = \bigcup_n \Delta_T^n$, where Δ_T^n $(n = 1, 2, 3, \cdots)$ are the maximal open connected components of Δ_T. In each connected component Δ_T^n, both nul $(T - bI)$ and def $(T - bI)$ are constant (and will be denoted by ν_n and μ_n, respectively), except for a finite or empty set Λ_T^n, formed by "exceptional" points, which are eigenvalues of T, whith finite algebraic multiplicities. For these points $\lambda \in \Lambda_T^n$, we have:

$$\text{nul}\,(T - \lambda I) = \nu_n + m_\lambda,$$

and

$$\text{def}\,(T - \lambda I) = \mu_n + m_\lambda,$$

where m_λ is the geometric multiplicity of λ.

If we define for $\alpha \in \mathbf{C}$ and $r \in \mathbf{R}^{+*}$,

$$B_r(\alpha) = \{x \in \mathbf{C} : |x - \alpha| < r\},$$

and by $B_r[\alpha]$, the set $\overline{B_r(\alpha)}$, we can state:

THEOREM. *Let* $T \in SF(X)$, Δ_T *be its semi-Fredholm domain, and* b *an element of* Δ_T, *such that* $\Delta(b) \cap \rho(T) \neq \varnothing$, *where* $\Delta(b)$ *is the maximal open connected component of* Δ_T, *containing* b.

If the following conditions hold:

1) There exists $\varepsilon > 0$, such that $B_\varepsilon[b] \subseteq \Delta(b)$ and $B_{\varepsilon+\delta}(b) \setminus \Delta(b) \neq \varnothing$, with $1/\delta = \max_{|\lambda - b| \leq \varepsilon} \|(T - \lambda I)^{-1}\|$,

2) There exists $\lambda \in (P_\sigma(T) \cap B_{\varepsilon+\delta}(b))$, where $P_\sigma(T)$ is the point spectrum set of T,

then, there is a $\xi \in (P_\sigma(T) \cap \Delta(b))$, such that:

(i) $X = E_\xi \oplus F_\xi$,

(ii) $0 < \dim E_\xi < \infty$,

(iii) $T(E_\xi) \subseteq E_\xi$ and $T(F_\xi) \subseteq F_\xi$,

(iv) $\sigma(T|_{E_\xi}) = \{\xi\}$ and $\sigma(T|_{F_\xi}) = \sigma(T) - \{\xi\}$.

Before proving the Theorem, we recall a result of Gohberg-Kreĭn [2], p. 4:

LEMMA. *Let* $T \in B(X)$ *and* $F \subseteq \rho(T)$, *with* F *closed in* $\mathbf{C}$; *then there exists a* $\delta > 0$ *such that: for all* $B \in B(X)$ *with* $\| B \| < \delta$, *we have* $F \subseteq \rho(T + B)$. *Moreover, the number* δ *is*:

$$\delta(T, F) = [\max_{\lambda \in F} \| (T - \lambda I)^{-1} \|]^{-1}. \tag{2.1}$$

We recall some simple consequences of the definition of $\delta(T, F)$:

1) $\delta \leq \| \xi I - T \|$,

2) $\delta(T - \beta I, -\beta + F) = \delta(T, F)$,

3) $\delta(\alpha T, \alpha F) = \alpha \delta(T, F)$.

for all $\xi \in F$, all $\alpha, \beta \in \mathbf{C}$, $T \in B(X)$ and $F \subseteq \rho(T)$, F being a closed set in $\mathbf{C}$.

Proof. of the theorem: If $\lambda \in \Delta(b)$, it's sufficient to take $\xi = \lambda$.

Suppose now, that $P_\sigma(T) \cap \Delta(b) = \varnothing$. Then we have, $\lambda \in (B_{\varepsilon+\delta}(b) \setminus \Delta(b))$.

Let $U = \beta \cdot (T - bI)$, where β is a positive real number, such that $\| T - bI \| < 1/\beta$, and let γ be the number $\beta \cdot (\lambda - b)$.

The number γ belongs to $P_\sigma(U)$. In addition, γ belongs to $B_{\beta(\varepsilon+\delta)}(0) \setminus B_{\beta\varepsilon}[0]$. So, we have that int $((-\gamma + B_{\beta\varepsilon}[0]) \cap B_{\beta\delta}(0))$ is a non-empty set.

Since $\| U \| < 1$, we conclude that $| \gamma | < 1$. Then, for every real positive number m, there are positive integers k_1 and k_2, such that:

$$e^{\gamma^{k_1}} \in B_m(1) \quad \text{and} \quad e^{\|U\|^{k_2}} \in B_m(1).$$

Consequently, it can be stated that there exist a positive integer k and and $\alpha \in \mathbf{C}$, such that

$$\alpha e^{\gamma^k} \in (-\gamma + B_{\beta\varepsilon}[0]) \quad \text{and} \quad \| \alpha e^{U^k} \| < \beta\delta.$$

But this leads us to a contradiction, because $(\gamma + \alpha e^{\gamma^k}) \in P_\sigma(U + \alpha e^{U^k})$ and by the Lemma above, $(\gamma + \alpha e^{\gamma^k}) \in \rho(U + \alpha e^{U^k})$, since $B_{\beta\varepsilon}[0] \subseteq \rho(U)$ and $\| \alpha e^{U^k} \| < \delta(U, B_{\beta\varepsilon}[0])$.

In conclusion, there exists $\xi \in P_\sigma(T) \cap \Delta(b)$, and, consequently, ξ has finite algebraic multiplicity. By Kato [1], p. 181 and 178, we obtain (i), (ii), (iii) and (iv).

An immediate and interesting consequence of this theorem is the following: if Δ is a maximal open connected component of Δ_T, with $\Delta \subseteq \rho(T)$, then,

$$\bigcup_{b \in \Delta} E(b) \cap P_\sigma(T) = \varnothing ,$$

where $E(b)$ is the union of all balls $B_{\varepsilon+\delta(\varepsilon)}(b)$, with $B[b] \subseteq \Delta$, and $\delta(\varepsilon)$ being the number $\delta(T, B_\varepsilon[b])$, defined by Eq. (2.1).

III. – Applications

In the following, we employ the usual notations and definitions for NFDE. For their description and for the general theory, we refer the reader to the recent books by Hale [3] and Izé [4].

Consider the linear periodic NFDE

$$\frac{d}{dt} D(t, x_t) = L(t, x_t), \tag{3.1}$$

with $D, L : \mathbf{R} \times C \to E^n$, continuous, linear in the second variable, D atomic at zero, at $\mathbf{R} \times C$, satisfying:

$$D(t + w, \varphi) = D(t, \varphi),$$

and

$$L(t + w, \varphi) = L(t, \varphi),$$

for some $w > 0$ and every $t \in \mathbf{R}^+$, $\varphi \in C$.

Associated with the equation (3.1), there is a class of linear bounded operators, called solution-operators of the Equation (3.1), which are defined by

$$T(t, s) : C \to C \cdot T(t, s)\varphi = x_t(s, \varphi) \qquad (t \geq s, \varphi \in C),$$

where $x_t(s, \varphi)$ is obtained from the solution, $x(t, s, \varphi)$, of the Equation (3.1).

The class $[T(t, s)]_{t \geq s}$ satisfies: $T(t + w, s) = T(t, s) T(s + w, s)$. Then, $T^n(w, 0) = T(nw, 0)$, for every positive integer n.

By the existence theorem for NFDE, and by the definition of $T(nw, 0)$, we have $T(nw, 0) \in SF(C)$. If $T(nw, 0)$ is $1 - 1$, for some positive integer n, then

$$0 \in \rho(T(nw, 0) \cap \Delta_{T(nw,0)}) .$$

Denoting by $\Delta(0)$ the open connected component of $\Delta_{T(nw,0)}$ containing zero, suppose that there exists a $\lambda \in P_\sigma(T(nw, 0)) \cap B_{\varepsilon+\delta}(0)$, where $B_\varepsilon[0] \subseteq \Delta(0)$ and $1/\delta$ is the number $\max\limits_{|\lambda| \leq \varepsilon} \|(\lambda I - T)^{-1}\|$. Applying the above theorem for $T(nw, 0) = T$ and $b = 0$, we conclude that exists a $\xi \in P_\sigma(T(nw, 0))$, such that

(i) $C = E_\xi \oplus F_\xi$,

(ii) $0 < \dim E_\xi < \infty$,

(iii) $T(E_\xi) \subseteq E_\xi$.

E_ξ is finite dimensional and, so, it has a basis formed by vectors $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_{d_\xi})$. Then by (iii) above, we see that there exists a non singular, $d_\xi \times d_\xi$ matrix, M, such that $T\Phi = \Phi M$.

PROPOSITION. *Let n be a positive integer, ε a positive real number and $\lambda \in P_\sigma(T(nw, 0))$, such that $T(nw, 0)$ is $1-1$, $B_\varepsilon[0] \subseteq \Delta(0)$, and $\lambda \in (\Delta(0) \cap B_{\varepsilon+\delta}(0))$, where $1/\delta = \max_{|\xi|\leq\varepsilon} \|(\xi I - T)^{-1}\|$. Then there exists a $n \times d_\xi$ matrix, $P(t)$, such that, for all $t \in R$, we have:*

(i) $P(t) = P(t + w)$,

(ii) $T(t, 0)\Phi = P\left(\frac{t}{n}\right) e^{Bt}$, *where* $B = \frac{\ln M}{nw}$

Proof: It is sufficient to show the existence of a P, satisfying (i) and (ii). Define P as

$$P(t) = T(nt, 0)\Phi e^{-Bnt}, \quad \text{for} \quad t \geq 0.$$

Then,

$$P(t+w) = T(nw + nt, 0)\Phi e^{-Bnw} e^{-Bnt} =$$
$$= T(nt, 0) T(nw, 0)\Phi e^{-Bnw} e^{-Bnt} = T(nt, 0)\Phi e^{-Bnt} = P(t).$$

If $t < 0$, it's sufficient to choose a positive integer k such that $t + kw > 0$, and to make $P(t)$ assume the value of P at the point $t + kw$. In this way, $P(t)$ is defined for all $t \in R$.

REMARK. Concerning the assumption that $T(nw, 0)$ be $1-1$, we observe a result for Retarded Differential Functional Equations, due to J. K. Hale and W. Oliva [5], stating that the class of linear RFDE with the solution-operators, $1-1$, is dense in the class of all linear RFDE.

Finally, for the class of linear bounded operators $[S(t)]_{t\geq 0}$, in X (X being a Banach space), verifying

(i) for all $t \geq 0$, $S(t) \in SF(X)$,

(ii) for some real positive number w, and all positive integer n,

$$S(nw) = S^n(w),$$

we have the following more general result: for every p, such that there exists a maximal open connected component of $\Delta_{S(pw)}$, Δ, with $\Delta \cap \rho(S(pw)) \neq \varnothing$

and for which the set

$$(3.2) \qquad \bigcup_{n:n|p} \left(P_\sigma (S(nw)) \cap \left[\bigcup_{b \in \Delta} E(b) \right]^{n/p} \right)$$

is nonempty, there exists an eigenvalue ξ of $S(pw)$, that decomposes X as

$$X = E_\xi \oplus F_\xi ,$$

with $0 < \dim E_\xi < \infty$.

$E(b)$, in Eq. (3.2) denotes the union of all balls $B_{\varepsilon+\delta(\varepsilon)}(b)$, such that $B[b] \subseteq \Delta$, being $\delta(\varepsilon)$ the number $\delta(S(pw), B_\varepsilon[b])$, defined by Eq. (2.1), for every $b \in \Delta$.

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