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Rendiconti

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Basis in a certain Completion of the s-d-ring over the ration al Numbers

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# RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 15 giugno 1978

Presiede il Presidente della Classe Antonio Carrelli

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Algebra. — *Basis in a certain Completion of the s-d-ring over the rational Numbers.* Nota II di Esayas George Kundert, presentata (*) dal Socio G. Zappa.

Riassunto. — Vedere il riassunto della Nota I apparsa nel precedente fascicolo di questi Rendiconti.

Example (2'). We know that $K^{-2} = E - D_2$ is a homomorphism of $\hat{\mathscr{A}}$ onto itself.

$\left\{y'_n = (-1)^n \sum_{k=0}^{\infty} \binom{n}{k} y'_k\right\}$ is then a K^{-2}-basis for the N_{D_2}-algebra $\hat{\mathscr{A}}$ and we have inversely $y_k = (-1)^k \sum_{n=0}^{\infty} \binom{n}{k} y'_n$. Let A_2 be the **Q**-algebra ΠN_{D_2} (direct product of infinitely many copies of N_{D_2}). Let Δ_2 be the mapping $\hat{\mathscr{A}} \to \hat{A}_2$ which associates to $a = \sum_{k=0}^{\infty} \alpha_k v'_k$ the element $\alpha = (\alpha_k)$, $\alpha_k \in N_{D_2}$, then we see—as in example (1')—that Δ_2 is an isomorphism between $\hat{\mathscr{A}}$ and $\hat{A}_2$ and we can define in $\hat{A}_2$ a semi-derivation d_2 by $d_2(\alpha) = \Delta_2 D_2 \Delta_2^{-1}(\alpha)$, so that we have again $d_2 \Delta_2 = \Delta_2 D_2$. The mapping $\Delta_{12} = \Delta_1 \Delta_2^{-1}$ is an isomorphism from $\hat{A}_2$ into $\hat{A}_1$ and we can define a new semi-derivation d_{12} in $\hat{A}_1$ by letting:

$$d_{12} = \Delta_{12} d_2 \Delta_{12}^{-1} = \Delta_1 \Delta_2^{-1} d_2 \Delta_2 \Delta_1^{-1} = \Delta_1 D_2 \Delta_1^{-1} = \Delta_1 D_1 (2 - D_1) \Delta_1^{-1} =$$
$$= \Delta_1 D_1 \Delta_1^{-1} \Delta_1 (2 - D_1) \Delta_1^{-1} = d_1 (2 - d_1) .$$

(*) Nella seduta del 13 maggio 1978.

Example (3'). Let $H' = E - DQ_1$. An H'-basis is then

$$\left\{u'_n = (-1)^n \sum_{k=0}^{\infty} \binom{k}{n} u_k\right\}.$$

In the $\{x_k\}$-basis we have $u'_n = (-1)^n \sum_{k=n+1}^{\infty} (-1)^k k C_{n+1}^k x_k$ and in the $\{x'_k\}$-basis we have $u'_n = (-1)^n \sum_{k=1}^{\infty} (-1)^k C_n^k x'_k$. Inversely

$$x_k = (-1)^k \sum_{n=0}^{\infty} B_{k+1,k}^n u'_n \qquad \text{and} \quad x'_k = \sum_{n=0}^{\infty} (-1)^n B_n^k u'_n$$

where C_k^n and B_n^k are again the Stirling numbers of the first and second kind respectively and $B_{k+1,k}^n$ are Nielsen numbers. (See [3])

We may ask: What is the series expansion of the square of u'_1?

Since

$$u'_1 = \sum_{k=1}^{\infty} (1/k)\, x'_k$$

and therefore

$$(u'_1)^2 = \sum_{k=1}^{\infty} (1/k^2)\, x'_k = \sum_{k=1}^{\infty} (1/k^2) \sum_{n=0}^{\infty} (-1)^n B_n^k u'_n = \sum_{n=1}^{\infty} (-1)^n \left(\sum_{k=1}^{n} (1/k^2)\, B_n^k\right) u'_n,$$

since

$$b_n = (-1)^n \sum_{k=1}^{n} B_n^k / k^2$$

are exactly the classical Bernoulli numbers, we have

$$(u'_1)^2 = \sum_{k=1}^{\infty} b_n u'_n$$

and this can be used as a new interpretation of the Bernoulli numbers. (See [1], p. 20).

Another interpretation of the Bernoulli numbers one obtains, if one represents the element $h = \sum_{k=0}^{\infty} (1/(k+1))\, x_k$ with respect to the H'-basis $\{u'_n\}$ because we have then

$$h = \sum_{k=0}^{\infty} (1/(k+1)) (-1)^k \sum_{n=0}^{\infty} B_{k+1,k}^n u'_n =$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} (-1^k B_{k+1,k}^n / k + 1\right) u'_n = \sum_{n=0}^{\infty} b_n u'_n .$$

We have therefore also the following formula for the Bernoulli numbers:

$$b_n = \sigma (H')^n h .$$

Remark. Since by [6], p. 487: $\sigma = SQ_{-1}$, we can also, whenever $\sigma_A = \sigma$, express the coefficients α_n, in the series expansion with respect to an A-basis, in the form of an integral with kernel Q_{-1} (generalized Fourier series!) namely $\alpha_n = SQ_{-1} A^n a$. In particular therefore also:

$$b_n = SQ_{-1} (H')^n h$$

Let $\{a_k\}$ be an A-basis and $\{b_k\}$ be a B-basis such that $N_A \subseteq N_B$. Now if

$$a = \sum_{k=0}^{\infty} \alpha_k a_k \qquad \text{let} \quad {}_A T_B (a) = \sum_{k=0}^{\infty} \alpha_k b_k .$$

${}_A T_B$ is linear and injective. If $N_A = N_B$ then ${}_A T_B$ is also surjective. In this case ${}_A T_B^{-1} = {}_B T_A$. If $B = A'$ and if $\{a_n'\}$ is an A'-basis the ${}_A T_{A'} (a_n) = a_n'$ and

$${}_A T_{A'} (a_n') = (-1)^n \sum_{k=0}^{\infty} (-1)^k \binom{k}{n} {}_A T_{A'} (a_k) = (-1)^n \sum_{k=0}^{\infty} (-1)^k \binom{k}{n} a_k' = a_n .$$

Therefore ${}_A T_{A'}^2 = E$ and ${}_A T_{A'}$ is a reflection. If ${}_A T_B$ and ${}_B T_C$ are defined then ${}_A T_C$ is also defined and ${}_A T_C = {}_B T_C \cdot {}_A T_B$.

We want to study the fixpoint set ${}_A F_{A'}$ *of the mapping* ${}_A T_{A'}$. *To investigate* ${}_A F_{A'}$ *we may employ the following principles*:

(1) *It is clear that* ${}_A F_{A'}$ *is a linear subspace of the* N_A*-algebra* $\hat{\mathscr{A}}$, *it is however not closed under the multiplication of* $\hat{\mathscr{A}}$.

(2) *Let* $a = \sum_{k=0}^{\infty} \alpha_k a_k$ *be a fixpoint. We must have*

$$\alpha_k = \sigma_A A^k a = \sigma_A A'^k a = \sigma_A (E - A)^k a = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \alpha_j .$$

We have therefore free choice in selecting the α_k *with* k *even, but then the* α_k *with* k *odd are uniquely determined by the condition*

$$(F)\ \alpha_k = 1/2 \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \alpha_j \qquad \textit{for } k \textit{ odd.}$$

Or we can choose the α_k *for* k *odd and determine the* α_k *for* k *even by solving* (F) *for* α_{k-1}.

(3) *From condition* (F) *above it follows at once that if* $a \in {}_A F_{A'}$ *then* ${}_A T_B (a) \in {}_B F_{B'}$ *in other words* ${}_A T_B ({}_A F_{A'}) \subseteq {}_B F_{B'}$ *and if* $N_A = N_B$ *then* ${}_A F_{A'}$ *and* ${}_B F_{B'}$ *are isomorphic linear subspaces. In particular* ${}_A F_{A'} = {}_{A'} F_A$.

(4) *Assertion*: ${}_D F_{D'} = {}_H F_{H'}$,

Proof. We only have to show that if $a \in {}_D F_{D'} \Rightarrow a \in {}_H F_{H'}$ or that from $\alpha_k = \sigma D^k a = \sigma D'^k a = \alpha_k' \Rightarrow \beta_k = \sigma H^k a = \sigma H'^k a = \beta_k'$.

Now $a = \sum_{k=0}^{\infty} \alpha_k x_k = \sum_{k=0}^{\infty} \alpha_k x'_k$ and we know $x_k = (-1)^k \sum_{n=0}^{\infty} B^n_{k+1,k} u'_n$ and therefore

$$a = \sum_{k=0}^{\infty} (-1)^k \alpha_k \sum_{n=0}^{\infty} B^n_{k+1,k} u'_n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} (-1)^k B^n_{k+1,k} \alpha_k \right) u'_n = \sum_{n=0}^{\infty} \beta'_n u'_n .$$

On the other hand we know that

$$x'_k = (-1)^k \sum_{n=0}^{\infty} B^n_{k+1,k} u_n$$

and therefore similarly

$$a = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} (-1)^k B^n_{k+1,k} \alpha_k \right) u_n = \sum_{n=0}^{\infty} \beta_n u_n$$

so that $\beta_n = \beta'_n$.

(5) *Let* $\mathrm{L} = \mathrm{DD}' = \mathrm{D}'\mathrm{D}$ *and if*

$$a = \sum_{k=0}^{\infty} \alpha_k x_k = \sum_{k=0}^{\infty} \alpha'_k x'_k \qquad \text{let} \quad \mathrm{S_D}(a) = \sum_{k=0}^{\infty} \alpha_k x_{k+1}$$

and $\mathrm{S_{D'}} = \sum_{k=0}^{\infty} \alpha'_k x'_{k+1}$ *and let* $\mathrm{S_L} = \mathrm{S_D}\,\mathrm{S_{D'}}$. *Note that* $\mathrm{LS_L} = \mathrm{E}$.

Assertion: *If* $a \in {}_{\mathrm{D}}\mathrm{F}_{\mathrm{D}'} \Rightarrow \mathrm{L}(a) \in {}_{\mathrm{D}}\mathrm{F}_{\mathrm{D}'}$ *and* $\mathrm{S_D}(a) \in {}_{\mathrm{D}}\mathrm{F}_{\mathrm{D}'}$.

Proof. $$\mathrm{D}(a) = \sum_{k=0}^{\infty} \alpha_k x_{k-1} = \sum_{k=0}^{\infty} \alpha_k (-1)^{k-1} \sum_{j=k-1}^{\infty} \binom{j}{k-1} x'_j =$$
$$= \sum_{j=0}^{\infty} \left[\sum_{k=1}^{j+1} (-1)^{k-1} \binom{j}{k-1} \alpha_k \right] x'_j .$$

Let

$$\beta'_j = \sum_{k=1}^{j+1} (-1)^{k-1} \binom{j+1}{k-1} \alpha_k \Rightarrow \mathrm{L}(a) = \sum_{j=0}^{\infty} \beta'_j x'_j .$$

Now by condition (F) we have

$$\alpha_k = 1/2 \sum_{h=0}^{k-1} (-1)^k \binom{k}{n} \alpha'_h \Rightarrow \beta'_j = 1/2 \sum_{k=1}^{j+1} (-1)^{k-1} \binom{j+1}{k-1} \sum_{h=0}^{k-1} (-1)^h \binom{k}{n} \alpha_h =$$
$$= 1/2 \sum_{h=0}^{j-1} (-1)^h \binom{j}{h} \beta'_h \Rightarrow \mathrm{L}(a) \in {}_{\mathrm{D}}\mathrm{F}_{\mathrm{D}'}$$

by condition (F) and similarly one proves $\mathrm{S_D}(a) \in {}_{\mathrm{D}}\mathrm{F}_{\mathrm{D}'}$:

Examples of elements in ${}_{\mathrm{D}}\mathrm{F}_{\mathrm{D}'}$:

(*a*) $0 \in {}_{\mathrm{D}}\mathrm{F}_{\mathrm{D}'}$, (*b*) $a = 2 + \sum_{n=1}^{\infty} x_n = 1 + x'_0 \in {}_{\mathrm{D}}\mathrm{F}_{\mathrm{D}'}$,

(c) $S_L\, a = 2\, x_2 + \sum_{n=3}^{\infty} n x_n \in {}_D F_{D'}$, ($d$) $h = \sum_{n=0}^{\infty} \frac{x_n}{n+1} \in {}_D F_{D'}$ see example (3′) from before.

(e) $b = {}_{H'}T_D\, h \in {}_D F_{D'}$ by the discussion at the end of example (3′) and the definition of ${}_{H'}T_D$ we see that the element b has the Bernoulli numbers b_n as coefficients in the series expansion of b with respect to the D-basis $\{x_n\} : b = \sum_{n=0}^{\infty} b_n x_n$. We have therefore a third interpretation of Bernoulli numbers, we can say:

The uniquely determined element $b \in {}_D F_{D'}$ which has the coefficients $b_0 = 1$, $b_{2n} = 0$, has the (nonzero) classical Bernoulli numbers as coefficients in the D-basis expansion of b.

In another article we will show, that we can use the three new interpretations of Bernoulli numbers to prove properties, and also to obtain some natural generalizations, of these numbers.

LITERATURE

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