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Vincenzo Capasso, Donato Fortunato

Stability results for a diffusive nonlinear deterministic epidemic model

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Equazioni a derivate parziali. — *Stability results for a diffusive nonlinear deterministic epidemic model* (*). Nota di VINCENZO CAPASSO e DONATO FORTUNATO, presentata (**) dal Socio C. MIRANDA.

RIASSUNTO. — Si stabilisce un risultato di stabilità asintotica per la soluzione di equilibrio di una equazione semilineare di evoluzione che descrive un modello epidemico.

In this paper we study the semilinear evolution equation (7); an existence theorem and an asymptotic stability result of the time independent equilibrium solution zero are stated. The proof of these results is mainly based on the tools developed in [1] and [2].

The evolution equation we study is the abstract formulation of the following deterministic epidemic model proposed in [1], which includes spatial diffusion of both susceptible and infective individuals:

$$(1) \qquad \begin{cases} \dfrac{\partial}{\partial t} s(x\,;t) = d_1 \Delta s(x\,;t) - a(v(\cdot\,;t))(x)\, s(x\,;t) - \mu s(x\,;t) \\[2mm] \dfrac{\partial}{\partial t} v(x\,;t) = d_2 \Delta v(x\,;t) + a(v(\cdot\,;t))(x)\, s(x\,;t) - \lambda v(x\,;t) \\[2mm] \text{with} \quad t > 0 \quad , \quad x \in \Omega \end{cases}$$

supplemented by suitable boundary and initial conditions (Δ is the Laplacian).

Here $s(x\,;t)$ and $v(x\,;t)$ respectively denote the density of susceptibles and the density of infectives at time t, and at point x of the habitat Ω (a bounded domain in $\mathbf{R}^n$ with smooth boundary). d_1 and d_2, the diffusion coefficients, are supposed to be positive constants.

In the interaction term $a(v)\,s$, the dependence upon the density of infectives occurs via a map $a : L^1(\Omega) \to L^\infty(\Omega)$ which obeys to some assumptions [1], the most significant of which are:

$$\text{(A.1)} \qquad \forall f \in L^1(\Omega) : \| a(f) \|_{L^\infty(\Omega)} \le c_1 \,; \qquad a(f) \ge 0 \quad \text{if} \quad f \ge 0 \quad \text{and} \quad a(0) = 0\,.$$

$$\text{(A.2)} \qquad \forall f, g \in L^1(\Omega) : \| a(f) - a(g) \|_{L^\infty(\Omega)} \le c_2 \| f - g \|_{L^1(\Omega)}$$

(c_1 and c_2 are positive constants).

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This kind of interaction allows us to take into account saturation effects, and also the possibility of some kind of "distant" interaction between susceptibles and infectives [1].

In the first equation the term $-\mu s$ takes into account immunization effects, while in the second equation the term $-\lambda v$ represents the removal of infectives due to recovery (with immunization), death, and isolation (λ and μ are positive constants).

Zero Neumann boundary conditions will be assumed to indicate an isolated habitat during the evolution of the epidemic phenomenon.

Moreover system (1) is supplemented by the initial conditions

$$s(x;0) = s_0(x) \geq 0 \quad ; \quad v(x;0) = v_0(x) \geq 0 , \qquad x \in \Omega \tag{2}$$

(not identically zero).

In order to write the abstract formulation of system (1), (2) with the quoted boundary conditions, we introduce the following Hilbert spaces:

$$X_1^\alpha = H^\alpha(\Omega) \quad , \quad \alpha \geq 0 \tag{3}$$

(the usual fractional order Sobolev space, whose scalar product and norm will be respectively denoted by $(\cdot \mid \cdot)_\alpha^1$ and $\| \cdot \|_\alpha^1$ ([3, Chap. 1]), and

$$X^\alpha = H^\alpha(\Omega) \times H^\alpha(\Omega) \tag{4}$$

with scalar product

$$(\boldsymbol{f} \mid \boldsymbol{g})_\alpha = (f_1 \mid g_1)_\alpha^1 + (f_2 \mid g_2)_\alpha^1 , \qquad \text{if} \quad \boldsymbol{f} = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} , \boldsymbol{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

and norm

$$\| \boldsymbol{f} \|_\alpha = \| f_1 \|_\alpha^1 + \| f_2 \|_\alpha^1 , \qquad \text{if} \quad \boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} .$$

The standard selfadjoint realization in $L^2(\Omega)$ of the differential operator $u \to -\Delta v$ with zero Neumann boundary conditions $\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$ will be denoted by B.

Let us set

$$A = \begin{pmatrix} d_1 B + \mu I & 0 \\ 0 & d_2 B + \lambda I \end{pmatrix} \tag{5}$$

where I is the identity operator in $L^2(\Omega)$. A results to be a selfadjoint and positive operator in $X^0 = L^2(\Omega) \times L^2(\Omega)$, with compact inverse.

We further denote by F the following operator

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in X^0 \to F(\boldsymbol{f}) = \begin{pmatrix} -a(f_2) f_1 \\ a(f_2) f_1 \end{pmatrix} . \tag{6}$$

With this in mind, our problem can be reformulated as the following semilinear autonomous evolution equation for $\boldsymbol{u}(t) \equiv \begin{pmatrix} s(\cdot\,;t) \\ v(\cdot\,;t) \end{pmatrix}$:

$$\frac{d}{dt}\boldsymbol{u}(t) = -\mathrm{A}\boldsymbol{u}(t) + \mathrm{F}(\boldsymbol{u}(t)), \qquad t > 0. \tag{7}$$

To be more precise we look for a map $\boldsymbol{u} : [0, +\infty[\to \mathrm{X}^0$ satisfying the following conditions:

(i) $\boldsymbol{u} \in \mathscr{C}([0, +\infty[, \mathrm{D}(\mathrm{A}^\alpha))$, $\alpha \in]0, 1[$;

(ii) $\boldsymbol{u} \in \mathrm{D}(\mathrm{A})$ for $t \in]0, +\infty[$ and $\mathrm{A}\boldsymbol{u} \in \mathscr{C}(]0, +\infty[, \mathrm{X}^0)$,

(iii) $\boldsymbol{u} \in \mathscr{C}^1(]0, +\infty[, \mathrm{X}^0)$;

(iv) $\boldsymbol{u}$ solves (7) in $]0, +\infty[$;

(v) $\boldsymbol{u}(0) = \boldsymbol{u}^0 \equiv \begin{pmatrix} s_0(\cdot) \\ v_0(\cdot) \end{pmatrix} \in \mathrm{D}(\mathrm{A}^\alpha)$

where $\mathrm{D}(\mathrm{A}^\alpha)$ is the domain of A^α equipped with the graph–norm.

Due to the results obtained in [1] and in [2], the following existence theorem can be stated.

THEOREM 1. *Problem* (7) *admits a unique solution satisfying* (i)-(v). *If the initial conditions are such that* $s_0(\cdot) \geq 0$, $v_0(\cdot) \geq 0$, *then* $s(\cdot\,;t) > 0$, $v(\cdot\,;t) \geq 0$ *at any time* $t \geq 0$.

From Theorem 1 it follows in particular that the positive cone $\mathrm{D}_+(\mathrm{A}^\alpha)$ of $\mathrm{D}(\mathrm{A}^\alpha)$ is an invariant set with respect to the evolution described by Eq. (7), (i.e. if $\boldsymbol{u}^0 \in \mathrm{D}_+(\mathrm{A}^\alpha)$, then $\boldsymbol{u}(t) \in \mathrm{D}_+(\mathrm{A}^\alpha)$ at any $t \geq 0$). Furthermore we can state that the only equilibrium solution of (7) in $\mathrm{D}_+(\mathrm{A}^\alpha)$ is the trivial solution $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Due to the results listed in [2], the following stability result can be proved.

THEOREM 2. *The trivial solution is* $\mathrm{D}(\mathrm{A}^\alpha) - \mathrm{D}(\mathrm{A}^\alpha)$ *asymptotically stable.*

This means that $\mathbf{0}$ is stable and attractive (i.e. for any $\varepsilon > 0$, a $\delta_\varepsilon > 0$ exists, such that if $\| \boldsymbol{u}^0 \|_{\mathrm{D}(\mathrm{A}^\alpha)} < \delta_\varepsilon$, then $\| \boldsymbol{u}(t) \|_{\mathrm{D}(\mathrm{A}^\alpha)} < \varepsilon$, and also $\lim_{t \to +\infty} \| \boldsymbol{u}(t) \|_{\mathrm{D}(\mathrm{A}^\alpha)} = 0$).

Moreover it can be shown that for any $\varepsilon > 0$, a $\delta_\varepsilon > 0$ exists such that if $\| \boldsymbol{u}^0 \|_{\mathrm{D}(\mathrm{A}^\alpha)} < \delta_\varepsilon$ then

$$\| \boldsymbol{u}(t) \|_{\mathrm{D}(\mathrm{A}^\alpha)} \leq \varepsilon e^{-bt} \qquad , \qquad t \in [0, +\infty[, \alpha \in]0, 1[. \tag{8}$$

Here $b \in]0, \sigma_0[$, where $\sigma_0 = \inf \sigma(\mathrm{A})$ if $\sigma(\mathrm{A})$ is the spectrum of A.

By well known results of interpolation theory [3, Chap. 1], it can be deduced that $\mathrm{D}(\mathrm{A}^\alpha)$ (equipped with the graph-norm of A^α) is continuously embedded into $X^{2\alpha}$; then, by the Sobolev embedding theorems, the following pointwise bound can be obtained $\left(\text{if } \frac{n}{4} < \alpha < 1\right)$:

$$\sup_{x \in \Omega} | \boldsymbol{u}(x;t) | \leq \varepsilon e^{-bt} \tag{9}$$

where ε and b are the same as before.

If $\alpha \in]3/4, 1[$ we have (e.g., see [3, p. 107])

$$D(A^\alpha) = \left\{ \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in X^{2\alpha} \;\middle|\; \left.\frac{\partial u_1}{\partial n}\right|_{\partial\Omega} = 0 \,,\, \left.\frac{\partial u_2}{\partial n}\right|_{\partial\Omega} = 0 \right\}$$

with the induced norm of $X^{2\alpha}$.

This, by virtue of what has been said above, allows us to obtain in this case the $X^{2\alpha} - X^{2\alpha}$ asymptotic stability.

Let us finally observe that $\sigma_0 = \inf \sigma(A) \geq \min(\mu, \lambda)$; this implies that the greater the minimum of μ and λ is, the greater b can be chosen in (9), improving the speed of decay of $\sup_{x \in \Omega} | \boldsymbol{u}(x;t) |$ to zero, which is "physically" reasonable.

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