Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Ethelberg N. Chukwu

On the Boundedness and the Existence of a Periodic Solution of Some Nonlinear Third Order Delay Differential Equation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **64** (1978), n.5, p. 440–447.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1978_8_64_5_440_0>

Equazioni differenziali ordinarie. — *On the Boundedness and the Existence of a Periodic Solution of Some Nonlinear Third Order Delay Differential Equation.* Nota di ETHELBERT N. CHUKWU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore utilizzando alcuni risultati di *T. Yoshizawa* prova che le soluzioni di una equazione differenziale non lineare del terzo ordine con argomento ritardato, sotto opportuna ipotesi sono uniformemente limitate e asintoticamente uniformente limitate e ne deduce l'esistenza di una soluzione periodica.

1. INTRODUCTION

In this paper we study the real third order delay differential equation of the form

$$(1) \qquad \dddot{x}(t) + f(x(t), \dot{x}(t), \ddot{x}(t))\,\ddot{x}(t) + g(x(t-h), \dot{x}(t-h)) + i(x(t-h)) = p(t, x(t), \dot{x}(t), x(t-h), \dot{x}(t-h), \ddot{x}(t))$$

in which f, g, i and p depend only on the arguments displayed explicitly and $h \geq 0$. Here the dots denote differentiation with respect to t. It is assumed as basic that $i'(x), g(x, y)$ and $p(t, x(t), y(t), x(t-h), y(t-h))$, are continuous in their respective arguments and that $\frac{\partial g}{\partial x}(x, y), \frac{\partial g}{\partial y}(x, y)$ $\frac{\partial f}{\partial x}(x, y, z)$ and $\frac{\partial f}{\partial z}(x, y, z)$ are continuous for all x, y, and z. We shall use the following notations. E^n is an n-dimensional linear real vector space with norm $|\cdot|$, and $C = C([-h, 0], E^n)$ is the space of continuous functions mapping $[-h, 0]$ into E^n with $\|\phi\| = \sup_{-h \leq \theta \leq 0} |\phi(0)|$ for $\phi \in C$. C_H will denote the set of $\phi \in C$ such that $\|\phi\| \leq H$. For any continuous function $x(u)$ defined on $-h \leq u < A, A > 0$ and any fixed $t, 0 \leq t \leq A$, the symbol x_t will denote the function in C defined by $x_t(\theta), -h \leq \theta \leq 0$.

Let $f(t, \phi)$ be a function defined on $[t_0\, \infty) x\ C_H$ with $f(t, \phi) \in E^n$, and let $\dot{x}(t)$ denote the right-hand derivative of $x(u)$ at $u = t$ consider the functional differential equation

$$(2) \qquad \dot{x}(t) = f(t, x_t).$$

DEFINITION 1. A function $x(t_0, \phi)$ is called a solution of (2) with initial condition $\phi \in C_H$ at time t_0 if there is an $A > 0$ such that $x(t_0, \phi)$ is a function

(*) Nella seduta del 13 maggio 1978.

from $[t_0 - h, t_0 + A]$ into E^n such that $x_t(t_0, \phi) \in C_H$ for $t_0 \leq t \leq t_0 + A$, $x_{t_0}(t_0, \phi) = \phi$ and $x(t_0, \phi)$ satisfies (2) for $t_0 \leq t < t_0 + A$.

DEFINITION 2. Let $V(t, \phi)$ be a continuous functional defined for $t \geq 0$, $\phi \in C_H$. The derivative of V along solutions of (2) will be denoted by $\dot{V}_{(2)}$ and is defined by the following relation

$$(3) \qquad \dot{V}_{(2)}(t, \phi) = \limsup_{\delta \to 0} (1/\delta) \{V(t + \delta, x_{t+\delta}(t_0, \phi)) - V(t, x_t(t_0, \phi))$$

where $x(t_0, \phi)$ is the solution of (2) with $x_{t_0}(t_0, \phi) = \phi$.

DEFINITION 3. We say that the solutions of (2) are uniformly bounded if for each $\alpha > 0$ there exists $\beta(\alpha)$ such that if $\|\phi\| \leq \alpha$ then $\|x_t(t_0, \phi)\| \leq \beta(\alpha)$. We say that the solutions of (2) are uniform ultimately bounded for bound β if there exists $\beta > 0$ and for each $\alpha > 0$ there exists $T(\alpha)$ such that if $\|\phi\| \leq \alpha$ we have $\|x_t(t_0, \phi)\| \leq \beta$ for $t \geq t_0 + T(\alpha)$.

Let S be the set of all $\phi \in C$ such that $\|\phi\| \geq R$ where $R > 0$ is a constant which may be large. To study the boundedness of solutions of (2) we state the following theorems of Yoshizawa [1].

THEOREM 1.1. *Suppose there exists a continuous functional* $V(t, \phi)$ *on* $[0, \infty) x S$ *which satisfies the following conditions*:

(i) $a(\|\phi\|) \leq V(t, \phi) \leq b(\|\phi\|)$, *where* $a(r)$, *is continuously increasing, positive for* $r > R$, $a(r) \to \infty$ *as* $r \to \infty$ *and* $b(r)$ *is continuous increasing*

(ii) $\dot{V}_{(2)}(t, \phi) \leq -c(\|\phi\|)$, *where* $c(r)$ *is continuous and positive for* $r > R$.

Then the solutions of (2) are uniformly bounded and uniform ultimately bounded.

THEOREM 1.2. *Suppose for any compact set* $K \leq C$, *there exists a constant* $L(K) > 0$ *such that*

$$|f(t, \phi) - f(t, \psi)| \leq L(K) \|\phi - \psi\|, \phi, \psi \in K$$

and that $f(t, \phi)$ *is periodic in* t *of period* w, $w \geqq h$ i.e. $f(t + w, \phi) = f(t, \phi)$. *If the solutions of* (2) *are uniform—bounded and uniform—ultimately bounded for bound* β, *there exists a periodic solution of* (2) *of period which is bounded by* β.

The equation (1) *can be rewritten as*

$$(4) \qquad \begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= z(t) \\ \dot{z}(t) &= -f(x(t), y(t), z(t))\, z(t) - g(x(t), y(t)) - i(x(t)) + \\ &\quad + \int_{-h}^{0} g_x(x(t+\theta), y(t+\theta))\, y(t+\theta)\, d\theta + \end{aligned}$$

$$+\int_{-h}^{0} g_y (x (t+\theta), y (t+\theta)) z (t+\theta) \,d\theta +$$

$$+\int_{-h}^{0} i_x (x (t+\theta) y (t+\theta) \,d\theta +$$

$$+ p (t), x (t), y (t), x (t-h), y (t-h), z (t)) .$$

We shall investigate (1) *in this form.*

2. Statements of Result

THEOREM 2.1. *Suppose that* $g(x, 0) = i(0) = 0$ *and that*

(i) *there are constants* a, a', b *and* b' *such that*

$$a' \geq f(x, y, z) > a > 0 \quad \text{for all} \quad x, y, z,$$

and

$$b' \geq \frac{g(x, y)}{y} \geq b > 0 \quad \text{for} \quad y \neq 0 \quad \text{and all } x;$$

(ii) $i(x)/x \geq \delta \; (x \neq 0)$ *and* $i'(x) \leq c$ *for all* x *where* $\delta > 0, c$, $0 < c < ab$ *are constants*;

(iii) $$y\left(\frac{\partial f}{\partial x}(x, y, 0)\right) \leq 0 \quad , \quad y\left(\frac{\partial f}{\partial z}(x, y, z)\right) \geq 0 ,$$

$$-L \leq \frac{\partial g}{\partial x}(x, y) \leq 0 \quad \text{for all} \quad x, y, z, L > 0,$$

(iv) $| p(t, x(t), y(t), x(t-h), y(t-h), z(t) | \leq M$ *for all* $t, x(t)$, $y(t), x(t-h), y(t-h), z(t)$ *where* M *is a finite constant.*

Then for sufficiently small $h > 0$ every solution (x_t, y_t, z_t) of (4) is uniformly bounded and uniform ultimately bounded

THEOREM 2.2. *Assume all the conditions of Theorem* 2.1. *Suppose*

$$p(t+w, x(t), y(t), x(t-h), y(t-h), z(t))$$
$$= p(t, x(t), y(t), x(t-h), y(t-h), z(t))$$

where $w \geq h$. *Then there exists a periodic solution of* (4) *of period* w.

3. PROOF OF THEOREM 2.1.

We now construct the following functional

(5) $$V(x_t, y_t, z_t) = V_1(x, y, z) + V_2(x, y, z) + V_3(x, y, z) + V_4(x_t, y_t, z_t)$$

where

(6) $$2V_1 = 2\int_0^x i(s)\,ds + 2\int_0^y sf(x, s, 0)\,ds + 2\delta_1 \int_0^y g(x, s)\,ds + $$
$$+ 2\delta_1 z^2 + 2yz + 2\delta_1 yi(x);$$

(7) $$2V_2 = \delta_2 bx^2 + 2a\int_0^x i(s)\,ds + (a^2 - \delta_2)y^2 + $$
$$+ 2\int_0^y g(x, s)\,ds + z^2 + 2a\delta_2 xy + 2\delta_2 xz + $$
$$+ 2ayz + 2yi(x);$$

(8) $$2V_3 = 2a\int_0^y sf(x, s, 0)\,ds - a^2 y^2$$

and $\delta_1 > 0$, $\delta_2 > 0$ are two constants chosen (as is possible since $ab > c > 0$) such that

(9) $$1/\alpha < \delta_1 < b/c,\ ab - c > a\delta_2 > 0,$$
$$\delta_1 > \{(b' - b) + (a^1 - a)\}\,\delta_2/4;$$

(10) $$V_4 = \frac{2v}{h}\int_{-h}^{0}\left\{\int_{\theta_1}^{0}[x^2(t+\theta) + y^2(t+\theta)]\,d\theta\right\}d\theta_1 + $$
$$+ \frac{3v}{h}\int_{-h}^{0}\left\{\int_{\theta_1}^{0} z^2(t+\theta)\,d\theta\right\}d\theta_1$$

where $v \geq 0$ (chosen subsequently).

Observe that

$$W \equiv V_1 + V_2 + V_3$$

is the Lyapunov function defined by Harrow in [2] and utilized by Tejumola [3] when dealing with the ordinary scalar differential equation

(11) $$\dddot{x}(t) + f(x, \dot{x}, \ddot{x})\,\ddot{x} + g(x, \dot{x}) + i(x) = p(t, x, \dot{x}, \ddot{x}).$$

Because of the analysis in [2] and [3] the following properties of V are easily deduced.

LEMMA 3.1. *Subject to the conditions of Theorem 2.1. there are constants* $D_1 > 0$, $D_2 > 0$, $D_3 > 0$, $D_4 > 0$ *such that for all* x_t, y_t, z_t,

$$V_4 + D_1 (x^2(t) + y^2(t) + z^2(t)) \leq V(x_t, y_t, z_t) \leq \leq V_4 + D_2 (x^2(t) + y^2(t) + z^2(t)) \tag{12}$$

and

$$\begin{aligned} \dot{V}_{(4)}(x_t, y_t, z_t) \leq & -4 D_4 (x^2(t) + y^2(t) + z^2(t)) + \\ & + D_3 [|x(t)| + |y(t)| + |z(t)|] \left[p + \int_{-h}^{0} i_x(x(t+\theta)) y(t+\theta) d\theta + \right. \\ & + \int_{-h}^{0} g_x(x(t+\theta), y(t+\theta)) y(t+\theta) d\theta + \\ & \left. + \int_{-h}^{0} g_y(x(t+\theta), y(t+\theta)) z(t+\theta) d\theta \right] + \\ & + \frac{2v}{h} \int_{-h}^{0} [x^2(t) - x^2(t+\theta) + y^2(t) - y^2(t+\theta)] d\theta + \\ & + \frac{3v}{h} \int_{-h}^{0} [z^2(t) - z^2(t+\theta)] d\theta . \end{aligned} \tag{13}$$

The estimate of $\dot{V}_{(4)}$ is sharpened in the next Lemma.

LEMMA 3.2. *Subject to the conditions of Theorem 2.1.*

$$\dot{V}_{(4)} \leq -D_6 < 0 ,$$

provided

$$x^2(t) + y^2(t) + z^2(t) \geq D_1 > 0 ,$$

Proof. Let

$$k = \max [D_3 M , D_3 (c + L) , D_3 b']$$

and define v as

$$\nu \overset{\text{def}}{=} \frac{D_4}{2} - \left(\frac{D_4^2 - h^2 k^2}{4} \right)^{1/2} \geq 0 \tag{14}$$

if $hk \leq D_4$.

Observe that this inequality will be satisfied as $h \to 0$. It is clear from (12) that

$$(15) \qquad \dot{V}_{(4)} \leq -4\,D_4\,[x^2(t) + y^2(t) + z^2(t)] + $$

$$+ D_3\,[|x(t)| + |y(t)| + |z(t)|] \cdot$$

$$\cdot \left[M + (c+L)\int_{-h}^{0} |y(t+\theta)|\,d\theta + b'\int_{-h}^{0} |z(t+\theta)|\,d\theta\right] +$$

$$+ \frac{2v}{h}\int_{-h}^{0}\{x^2(t) - x^2(t+\theta) + y^2(t) - y^2(t+\theta)\}\,d\theta +$$

$$+ \frac{3v}{h}\int_{-h}^{0}\{z^2(t) - z^2(t+\theta)\}\,d\theta\,.$$

This inequality can in turn be improved and recast in the following form

$$(16) \qquad \dot{V}_{(4)} \leq -2\,D_4(x^2(t) + y^2(t)) - D_4\,z^2(t) + L\,[|x(t)| + |y(t)| + z(t)|] +$$

$$-\frac{2v}{h}\int_{-h}^{0} x^2(t+\theta)\,d\theta - 2\,\mu z^2(t) +$$

$$-\frac{1}{h}\int_{-h}^{0}\{\mu x^2(t) + vy^2(t+\theta) - kh\,|x(t)|\,y(t+\theta)|\}\,d\theta +$$

$$-\frac{1}{h}\int_{-h}^{0}\{\mu x^2(t) + vz^2(t+\theta) - kh\,|x(t)||z(t+\theta)|\}\,d\theta +$$

$$-\frac{1}{h}\int_{-h}^{0}\{\mu y^2(t) + vy^2(t+\theta) - kh\,|y(t)||y(y+\theta)|\}\,d\theta +$$

$$-\frac{1}{h}\int_{-h}^{0}\{\mu y^2(t) + vz^2(t+\theta) - kh\,|y(t)||z(t+\theta)|\}\,d\theta +$$

$$-\frac{1}{h}\int_{-h}^{0}\{\mu z^2(t) + vz^2(t+\theta) - kh\,|z(t)||z(t+\theta)|\}\,d\theta$$

where

$$\mu = \frac{D\,4}{2} + \left(\frac{D\,4^2 - k^2 h^2}{4}\right)^{1/2} > 0\,.$$

Each of the five integrands enclosed in the curvly bracket $'\{\cdot\}$ is positive semi definite. Hence

$$\dot{V}_{(4)} \leq - D_4 (x^2(t) + y^2(t) + z^2(t) - \frac{2v}{h} \int_{-h}^{0} x^2(t+\theta) + \tag{17}$$

$$+ k [| x(t) | + | y(t) | + | z(t) |] .$$

Now set

$$\xi(t) = \max [| x(t) | , | y(t) | , | z(t) |] .$$

If $\xi(t) = | x(t) |$ then (at least)

$$\dot{V}_{(4)} \leq - D_4 x^2(t) + 3 | x(t) | - \frac{2v}{h} \int_{-h}^{0} x^2(t+\theta) \, d\theta \leq$$

$$\leq - \frac{D_4}{2} x^2(t) - \frac{2v}{h} \int_{-h}^{0} x^2(t+\theta) \, d\theta$$

provided $| x(t) | \geq D_5 > 0$ where D_6 depends on D_4 and k.

Thus in case $\xi(t) = | x(t) |$ we have

$$\dot{V}_{(4)} \leq - \frac{D_4}{2} D_5^2 < 0$$

provided

$$| x(t) | \geq D_5 > 0 .$$

The same conclusion is true for

$$\xi(t) = | y(t) | \quad \text{and} \quad \xi(t) = | z(t) | .$$

Hence

$$\dot{V}_{(4)} \leq - D_7 < 0 ,$$

provided

$$x^2(t) + y^2(t) + z^2(t) \geq R ,$$

for some $D_1 > 0$ and some $R > 0$.

This proves the Lemma.

We now make use of (12) and Lemma 3.2 to complete the proof of Theorem 2.1. Indeed by (12) and Lemma 3.2 it is true that conditions (i) and (ii) of Theorem 1.1 are satisfied for the functional defined for (4) in (5). Uniform boundedness and uniform ultimate boundedness now follow for the solutions of 14. Theorem 2.2 follows immediately from Theorem 1.2.

REFERENCES

[1] T. YOSHIZAWA (1964) – *Ultimate Boundedness of Solutions and Periodic Solution of Functional. Differential Equations*, Colloques Internation aux sur les Vibrations Forcées dans les Systéms Nonlineaires. Marseille, September, 167–179.

[2] M. HARROW (1968) – « J. London Math Soc. », *43*, 587–192.

[3] H. O. TEJUMOLA – *A note on the Boundedness and the Stability of Solutions of Certain Third Order Differential Equations*, « Ann. Mat. Pura Appl. », (iv), *92*, 65–75.