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Nonoscillatory generating delay terms in functional differential inequalities

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Equazioni differenziali ordinarie. — *Nonoscillatory generating delay terms in functional differential inequalities* (*). Nota (**) di LU-SAN CHEN e CHEH-CHIH YEH, presentata dal Socio G. SANSONE.

RIASSUNTO. — Questa Nota estende precedenti risultati di altri e degli stessi Autori ad alcune classi di disequazioni differenziali.

INTRODUCTION

This note extends and improves the results of the authors [*Nonoscillation generating delay terms in odd order differential equations*, "Rend. Accad. Sci. Fis. Mat. Napoli" *43* (1976), 189–198] and Dahiya's [*Nonoscillation generating delay terms in even order differential equations*, "Hiroshima Math. J.", 5 (1975), 385–394] to the more general differential inequalities of the form

$$x(t)\left\{L_n x(t) + \delta \sum_{i=1}^{m} p_i(t) f_i(x[g_{i1}(t)], \cdots, x[g_{ik}(t)]) - h(t)\right\} \leq 0, \quad n: \text{odd}$$

and

$$x(t)\left\{L_n x(t) + \delta \sum_{i=1}^{m} p_i(t) f_i(x[g_{i1}(t)], \cdots, x[g_{ik}(t)]) - h(t)\right\} \geq 0, \quad n: \text{even}$$

where L_n is an operator defined by

$$L_0 x(t) = x(t), \quad L_\varkappa x(t) = \frac{1}{r_\varkappa(t)} (L_{\varkappa-1} x(t))', \quad r_n(t) = 1,$$
$$\delta = \pm 1, \quad \varkappa = 1, 2, \cdots, n.$$

Recently, Dahiya [2] and the authors [1] discussed the non-oscillatory property of solutions of the following equations

$$x^{(2n)}(t) - \sum_{i=1}^{m} p_i(t) x[g_i(t)] = h(t),$$

$$x^{(2n+1)}(t) + \sum_{i=1}^{m} p_i(t) x[g_i(t)] = h(t).$$

The purpose of this note is to improve and extend their results to the more general differential inequalities of the form:

(A) $\qquad x(t) M[x(t), \delta] \leq 0, \qquad$ for n odd

(B) $\qquad x(t) M[x(t), \delta] \geq 0, \qquad$ for n even,

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where

$$M[x(t), \delta] = L_n x(t) + \delta \sum_{i=1}^{m} p_i(t) f_i(x[g_{i1}(t)], \cdots, x[g_{ik}(t)]) - h(t).$$

Here L_n is an operator defined by

$$L_0 x(t) = x(t), \quad L_\varkappa x(t) = \frac{1}{r_\varkappa(t)} (L_{\varkappa-1} x(t))', \quad r_n(t) = 1, \quad \delta = \pm 1$$

for $\varkappa = 1, 2, \cdots, n$.

Throughout this note, we assume that the following conditions always hold:

(*a*) $p_i, g_{ij}, h, r_\varkappa \in C[R_+ \equiv [0, \infty), R_+ \setminus \{0\}], g_{ij}(t) \leq t$,

$\lim_{t\to\infty} g_{ij}(t) = \infty, g_{ij}(t)$ *is nondecreasing*, $g(t) = \max_{\substack{1\leq i\leq m \\ 1\leq j\leq k}} g_{ij}(t)$,

and

$$\int^{\infty} r_\varkappa(t)\, dt = \infty \text{ for } i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, k;\ \varkappa = 1, 2, \cdots, n-1.$$

(*b*) $f_i \in C[R^k, R \equiv (-\infty, \infty)]$ *is nondecreasing and for each*

$$y_j > 0 \qquad (j = 1, 2, \cdots, k)$$

$$0 < f_i(y_1, y_2, \cdots, y_k) \leq -f(-y_1, -y_2, \cdots, -y_k).$$

Moreover,

$$f(y_1, y_2, \cdots, y_k) = \min_{1\leq i\leq m} f_i(y_1, y_2, \cdots, y_k)$$

$$\liminf_{(\min y_j)\to\infty} f(y_1, y_2, \cdots, y_k) \geq 0 \quad \text{and} \quad \limsup_{(\max y_j)\to\infty} f(y_1, y_2, \cdots, y_k) < 0$$

where $(y_1, y_2, \cdots, y_k) \in R^k$.

A function is called *oscillatory* for $t \geq t_0 (> 0)$ if it has arbitrary large zeros. Otherwise it is called *nonoscillatory.*

The following lemma is an analog of a result due to Lovelady [4], so we omit the details.

LEMMA 1. *If $u(t)$ is a bounded positive function for $t \geq t_0$ such that $L_n u(t) > 0$ for n even or $L_n u(t) < 0$ for n odd, then $(-1)^\varkappa L_\varkappa u(t) > 0$ for $\varkappa = 0, 1, \cdots, n$.*

THEOREM 1. *Suppose that*

$$(C_1) \qquad \limsup_{t\to\infty} \int_{g(t)}^{t} r_1(s_{n-1}) \int_{s_{n-1}}^{t} \cdots \int_{s_1}^{t} \sum_{i=1}^{m} p_i(s)\, ds\, ds_1 \cdots ds_{n-1}$$

$$\geq \limsup_{z\to\infty} \frac{z}{f(z, \cdots, z)}.$$

Then, if $\delta = +1$, *every bounded nonoscillatory solution of* (A) *is eventually positive, while if* $\delta = -1$, *every bounded nonoscillatory solution of* (B) *is eventually negative.*

Proof. The proof will only be given for (A). Suppose to the contrary that $x(t)$ is an eventually negative solution of (A) for $t \geq t_0$. Then

$$\text{(1)} \qquad \mathrm{M}[x(t), 1] \geq 0 .$$

Let $t_1 \geq t_0$ be large enough so that $x[g_i(t)] < 0$ for $t \geq t_1$, $i = 1, 2, \cdots, m$. By (1) and conditions (a), (b) we have $\mathrm{L}_n x(t) > 0$. By Lemma 1 for $\varkappa = 0, 1, \cdots, n$

$$\text{(2)} \qquad (-1)^{\varkappa+1} \mathrm{L}_\varkappa x(t) > 0 \qquad \text{for } t \geq t_1 .$$

If $s \leq t$, then $g_i(s) \leq g_i(t) \leq g(t)$. By (2)

$$x[g_i(s)] \leq x[g_i(t)] \leq x[g(t)] .$$

From (1),

$$\mathrm{L}_n(-x(t)) - \sum_{i=1}^{m} p_i(t) f_i(x[g_{i1}(t)], \cdots, x[g_{ik}(t)]) \leq -h(t) < 0 ,$$

which by condition (b) implies

$$\text{(3)} \qquad \mathrm{L}_n(-x(t)) + p_i(t) f_i(-x[g_{i1}(t)], \cdots, -x[g_{ik}(t)]) < 0 .$$

Let $y(t) = -x(t)$. Integrating (3) n-times we have

$$y(t) - y[g(t)] + (-1)\mathrm{L}_1 y(t) \int_{g(t)}^{t} r_1(s_{n-1})\, \mathrm{d}s_{n-1} + \cdots$$

$$+ (-1)^{n-1} \mathrm{L}_{n-1} y(t) \int_{g(t)}^{t} r_1(s_{n-1}) \int_{s_{n-1}}^{t} \cdots \int_{s_2}^{t} r_{n-1}(s_1)\, \mathrm{d}s_1\, \mathrm{d}s_2 \cdots \mathrm{d}s_{n-1}$$

$$+ f(x[g(t)], \cdots, x[g(t)]) \int_{g(t)}^{t} r_1(s_{n-1}) \int_{s_{n-1}}^{t} \cdots \int_{s_1}^{t} \sum_{i=1}^{m} p_i(s)\, \mathrm{d}s\, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1} < 0 ,$$

which implies

$$\text{(4)} \qquad y[g(t)] \geq y(t) + f(y[g(t)], \cdots, y[g(t)]) \int_{g(t)}^{t} r_1(s_{n-1}) \int_{s_{n-1}}^{t} \cdots$$

$$\int_{s_1}^{t} \sum_{i=1}^{m} p_i(s)\, \mathrm{d}s\, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1} .$$

Since $y'(t) < 0$ and $y(t) > 0$ for $t \geq t_1$, it follows that $y(t)$ and $y[g(t)]$ decrease to a limit $c\ (\geq 0)$ as $t \to \infty$. From (4) we have $c = 0$, and

$$\frac{y[g(t)]}{f(y[g(t)], \cdots, y[g(t)])} \geq \int_{g(t)}^{t} r_1(s_{n-1}) \int_{s_{n-1}}^{t} \cdots \int_{s_1}^{t} \sum_{i=1}^{m} p_i(s)\, ds\, ds_1 \cdots ds_{n-1} \tag{5}$$

for $t \geq t_1$. Taking the limit superior as $t \to \infty$ of both sides of (5), we obtain a contradiction to condition (C_1).

COROLLARY 1. *Let condition* (C_1) *hold. Then every bounded nonoscillatory solution of* $M[x(t), 1] = 0$ *is eventually positive, and every bounded nonoscillatory solution of* $M[x(t), -1] = 0$ *is eventually negative.*

REMARK 1. Let $r_\varkappa(t) = 1$, $\varkappa = 1, 2, \cdots, n$. Then condition (C_1) implies

$$\limsup_{t \to \infty} \int_{g(t)}^{t} [s - g(t)]^{n-1} \sum_{i=1}^{m} p_i(s)\, ds > (n-1)! \limsup_{z \to \infty} \frac{z}{f(z, \cdots, z)} \,. \tag{6}$$

REMARK 2. Let $r_\varkappa(t) = 1$, $\varkappa = 1, 2, \cdots, n$. If

$$\limsup_{t \to \infty} \int_{g(t)}^{t} \sum_{i=1}^{m} [g_i(t) - g_i(s)]^n p_i(s)\, ds > n! \tag{7}$$

Then, by the Authors [1] and Dahiya [2], the conclusion of Theorem 1 holds.

The following example shows that in some cases (6) is better than (7).

Example. Consider the delay differential equation

$$x^{(iv)}(t) - \frac{\alpha}{t^4} x(\sqrt{t}) = e^{-t}, \tag{8}$$

where α is a positive constant. It has no eventually positive solution, as follows from Theorem 1, since

$$\limsup_{t \to \infty} \int_{\sqrt{t}}^{t} \frac{\alpha}{s^4} [s - \sqrt{t}]^3\, dt = \infty .$$

On the other hand

$$\limsup_{t \to \infty} \int_{\sqrt{t}}^{t} \frac{\alpha}{s^4} [\sqrt{t} - \sqrt{s}]^3\, ds = \frac{\alpha}{3} ,$$

so that the criterion (7) is applicable to (8) only when $\alpha > 18$.

LEMMA 2. (Kusano and Onose [3]). *Let*

$$u'(t) - \frac{\rho'(t)}{\rho(t)} u(t) + \frac{\rho'(t)}{\rho(t)} \phi(t) = 0 , \tag{9}$$

where

$$\phi(t) \in C[[T, \infty), R], \quad \rho(t) \in C^1[[T, \infty), R]$$

and

$$\rho(t) > 0, \quad \rho'(t) > 0, \lim_{t\to\infty} \rho(t) = \infty.$$

Suppose that $u(t)$ *is a solution of* (9) *on* $[T, \infty)$ *satisfying* $u(T) = 0$. *If* $\lim_{t\to\infty} |\phi(t)| = \phi^*(t)$ *exists in the extended real line* R^*, *then* $\lim_{t\to\infty} |u(t)| = u^*(t)$ *exists in* R^*. *In particular,* $\phi^*(t) = \infty$ *implies* $u^*(t) = \infty$.

THEOREM 2. *Let condition* (C_1) *and* $\lim_{t\to\infty} w_\varkappa(t) = \infty$, $\varkappa = 1, 2, \cdots, n-1$

$$(C_2) \qquad \int^{\infty} w_{n-1}(t) \sum_{i=1}^{m} p_i(t)\, dt = \infty$$

$$(C_3) \qquad \int^{\infty} w_{n-1}(t)\, h(t)\, dt < \infty$$

holds, where $w_{n-1}(t)$ *is defined by*

$$w_0(t) = 1, w_\varkappa(t) = \int_{t_0}^{t} r_\varkappa(s)\, w_{\varkappa-1}(s)\, ds, \qquad k = 1, 2, \cdots, n-1.$$

Then every bounded nonoscillatory solution of (A) or (B) approaches zero as $t \to \infty$.

Proof. We only consider the case (A). Let $x(t)$ be a bounded nonoscillatory solution of (A). By Theorem 1, $x(t)$ and $x[g_{ij}(t)]$, $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, k$ are positive for $t \geq t_0$. But for $t \geq t_0$ the inequality (A) implies

$$(10) \qquad L_n x(t) + \sum_{i=1}^{m} p_i(t)\, f_i(x[g_{i1}(t)], \cdots, x[g_{ik}(t)]) \leq h(t).$$

We put

$$u_\varkappa(t) = \int_{t_0}^{t} w_{n-\varkappa}(s)\, L'_{n-\varkappa} x(s)\, ds, \qquad \varkappa = 0, 1, \cdots, n-1.$$

An integration by parts yields

$$u_\varkappa(t) = w_{n-\varkappa}(t)\, L_{n-\varkappa} x(t) - w_{n-\varkappa}(t_0)\, L_{n-\varkappa} x(t_0)$$
$$- \int_{t_0}^{t} r_{n-\varkappa}(s)\, w_{n-\varkappa-1}(s)\, L_{n-\varkappa} x(s)\, ds$$

$$= \frac{w_{n-\varkappa}(t)}{r_{n-\varkappa}(t)\, w_{n-\varkappa-1}(t)} w_{n-\varkappa-1}(t)\, L'_{n-\varkappa-1}\, x(t)$$

$$- w_{n-\varkappa}(t_0)\, L_{n-\varkappa}\, x(t_0) - \int_{t_0}^{t} w_{n-\varkappa-1}(s)\, L'_{n-\varkappa-1}\, x(s)\, ds$$

$$= \frac{w_{n-\varkappa}(t)}{w'_{n-\varkappa}(t)} u'_{\varkappa}(t) - u_{\varkappa}(t) - w_{n-\varkappa}(t_0)\, L_{n-\varkappa}\, x(t_0)\,,$$

which shows that $u_{\varkappa}(t)$ satisfies the differential equation

$$(11) \qquad u'(t) - \frac{w'_{n-\varkappa}(t)}{w_{n-\varkappa}(t)} u(t) + \frac{w'_{n-\varkappa}(t)}{w_{n-\varkappa}(t)} \phi_{\varkappa}(t) = 0\,,$$

where $\phi_{\varkappa}(t) = -u_{\varkappa-1}(t) - w_{n-\varkappa}(t_0)\, L_{n-\varkappa}(t_0)$.

Since $u_{\varkappa}(t_0) = 0$, $w_{n-\varkappa}(t) > 0$, $w'_{n-\varkappa}(t) > 0$ and $\lim_{t\to\infty} w_{n-\varkappa}(t) = \infty$ by Lemma 2, we conclude that $\lim_{t\to\infty} |u_{\varkappa}(t)|$ exists in R^* whenever $\lim_{t\to\infty} |u_{\varkappa-1}(t)|$ exists in R^*.

Multiplying (10) by $w_{n-1}(t)$ and integrating from t_0 to t, we have

$$(12) \quad \int_{t_0}^{t} w_{n-1}(s)\, L'_{n-1}\, x(s)\, ds + \sum_{i=1}^{m} \int_{t_0}^{t} w_{n-1}(s)\, p_i(s) f_i(x[g_{i1}(s)], \cdots, x[g_{ik}(s)])\, ds$$

$$\leq \int_{t_0}^{t} w_{n-1}(s)\, h(s)\, ds\,.$$

Suppose

$$(13) \qquad \sum_{i=1}^{m} \int_{t_0}^{\infty} w_{n-1}(t)\, p_i(t)\, f_i(x[g_{i1}(t)], \cdots, x[g_{ik}(t)])\, dt = \infty.$$

From (12) and condition (C_3) we have $\lim_{t\to\infty} u_0(t) = -\infty$. From Lemma 2 applied to (11) with $\varkappa = 1$ it follows that $\lim_{t\to\infty} |u_1(t)| = \infty$. Applying Lemma 2 again to (11) with $\varkappa = 2$, we have $\lim_{t\to\infty} |u_2(t)| = \infty$. Repeating in the same way, we arrive at $\lim_{t\to\infty} |u_{n-1}(t)| = \infty$, which implies $\lim_{t\to\infty} x(t) = \infty$, a contradiction to the boundedness of $x(t)$. Hence (13) is impossible. Thus

$$\sum_{i=1}^{m} \int_{t_0}^{\infty} w_{n-1}(t)\, p_i(t)\, f_i(x[g_{i1}(t)], \cdots, x[g_{ik}(t)])\, dt < \infty\,.$$

Letting $t \to \infty$ in (8), we see that $\lim_{t\to\infty} u_0(t)$ is finite. From Lemma 2 it follows that $\lim_{t\to\infty} u_1(t)$ exists in R^*. This limit must be finite, since otherwise we would be led to a contradiction to be boundedness of $x(t)$ as before. Continuing in this way, we find that $\lim_{t\to\infty} u_{n-1}(t)$ exists and is finite. This shows that $\lim_{t\to\infty} x(t)$ exists and is finite. On the other hand, using (*b*) and condition (C_2), we have

$$\liminf_{t\to\infty} x(t) = \liminf_{t\to\infty} y[g_{ij}(t)] = 0 \quad i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, k.$$

Therefore $\lim_{t\to\infty} y(t) = 0$.

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