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The structure of a class of Banach algebras generated by a normed space

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Analisi funzionale. — *The structure of a class of Banach algebras generated by a normed space* (*). Nota di SANDRO LEVI, presentata (**) dal Corrisp. G. CIMMINO.

RIASSUNTO. — Dato uno spazio normato M, si caratterizza la classe delle algebre di Banach commutative che sono generate da M e tali che ogni funzionale lineare su M di norma minore o uguale a uno si possa estendere ad un carattere dell'algebra stessa.

1. INTRODUCTION

Let A be a commutative complex Banach algebra with identity and let M be a subspace of A.

We say that M has the multiplicative extension property (m.e.p.) in A if every non-zero linear functional on M of norm at most one is the restriction to M of a multiplicative linear functional (m.l.f.) on A.

In (2) we have studied the subspaces with the m.e.p. of a given algebra.

We intend here to reverse our point of view: given a normed space M we will investigate the structure of a commutative Banach algebra with identity A which has the following three properties:

i) A contains (an isometric copy of) M;

ii) M has the m.e.p. in A;

iii) the algebra generated by M is dense in A.

Let us remark that condition ii) amounts to specifying the maximal ideal space of A since M has the m.e.p. in A if and only if the restriction map defined on the maximal ideal space of A with values in the closed unit ball of the dual of M (with the w^*-topology) is a surjective homeomorphism.

We recall the following results of (2):

α) (Theorem 2.5 and Remark). Let M be a subspace of A and A (M) the closed subalgebra generated by M. A linear functional L on M can be extended to a m.l.f. on A (M) if and only if for every finite subset $\{x_1, \cdots, x_n\}$ of M and every complex polynomial p in n variables the following inequality holds:

$$|p[L(x_1), \cdots, L(x_n)]| \leq \| p(x_1, \cdots, x_n) \|$$

(*) Lavoro eseguito nell'ambito del G.N.A.F.A. (C.N.R.).
(**) Nella seduta del 13 maggio 1978.

β) (Theorem 2.6.). If the subspace M has the m.e.p. in the algebra A ad S is any linearly independent subset of M, then S is also algebraically independent.

II. Algebraic results

Let A be a commutative complex algebra with identity; let M be a subspace of A and A_M the subalgebra generated by M and the identity. Let $\{x_i : i \in I\}$ be a Hamel basis for M and $\mathbf{C}[(X_i)]$ the algebra of complex polynomials in the set of indeterminates $\{X_i : i \in I\}$.

We shall say that M is algebraically independent if every linearly independent subset of M is algebraically independent.

LEMMA. M *is algebraically independent if and only if* A_M *is isomorphic to* $\mathbf{C}[(X_i)]$.

Proof. Let $v : \mathbf{C}[(X_i)] \to A_M$ be defined by $v[p((X_i))] = p((x_i))$ for every $p \in \mathbf{C}[(X_i)]$. Then v is a surjective homomorphism and $\ker v = \{p \in \mathbf{C}[(X_i)] : p((x_i)) = 0\}$.

If M has a basis which is algebraically independent then any linearly independent subset of M is algebraically independent and the lemma follows.

By the remark following Theorem 2.6 of (2) we know that if A is a Banach algebra the algebraic independence of M is not a sufficient condition for M to have the m.e.p. in A (M).

We can nevertheless prove a similar result in a purely algebraic setting.

Let M* be the algebraic dual of M.

We shall say that M has the algebraic m.e.p. in A_M if for every $f \in M^*$ there exists a m.l.f. on A_M which extends f.

PROPOSITION. M *has the algebraic* m.e.p. *in* A_M *if and only if* M *is algebraically independent.*

Proof. Suppose M is not algebraically independent. Then there exists $p \in \mathbf{C}[(X_i)]$, $p \not\equiv 0$, such that $p((x_i)) = 0$.

Since $p \not\equiv 0$ we can find $(a_i) \in \bigoplus_{i \in I} \mathbf{C}_i$ ($\mathbf{C}_i = \mathbf{C}\ \forall i \in I$) with $p((a_i)) \neq 0$. Let $f \in M^*$ be such that $f(x_i) = a_i\ \forall i \in I$.

If f could be extended to a m.l.f. h on A_M we would have:

$$0 = h[p((x_i))] = p((h(x_i)) = p((a_i)) \neq 0 .$$

Hence M does not have the algebraic m.e.p.

Suppose conversely that M is algebraically independent and let $f \in M^*$. Define h on A_M by $h[p((x_i))] = p((f(x_i)))$.

Then h is well defined, linear, multiplicative and extends f.

III. THE STRUCTURE OF THE ALGEBRAS GENERATED BY A NORMED SPACE

Let M be a normed space, A a Banach algebra verifying conditions i), ii) and iii), and $\{x_i : i \in \mathrm{I}\}$ a set of linearly independent elements of M whose span is dense in the completion of M.

Let $\mathrm{A_I}$ be the algebra generated in A by the x_i's and the identity and let $\mathbf{C}\,[(\mathrm{X}_i)]$ be as in II.

By β) and Lemma 1 $\mathrm{A_I}$ and $\mathbf{C}\,[(\mathrm{X}_i)]$ are isomorphic and it then follows from condition iii) that A is the completion of $\mathbf{C}\,[(\mathrm{X}_i)]$ for a norm which is consistent with conditions i) and ii).

It is a consequence of α) that a norm $\|\cdot\|_1$ on A is consistent with condition ii) if and only if for every $p \in \mathbf{C}\,[(\mathrm{X}_i)]$, $p = p\,(\mathrm{X}_1, \cdots, \mathrm{X}_n)$, we have

$$* \qquad \|p\,(x_1, \cdots, x_n)\|_1 \geq \sup |p\,[\mathrm{L}\,(x_1), \cdots, \mathrm{L}\,(x_n)]|$$

where L ranges over the closed unit ball of the dual of E.

In conclusion we have the following

THEOREM. *It is always possible to construct an algebra* A *which has properties i), ii) and iii). Any such* A *is the completion of* $\mathbf{C}\,[(\mathrm{X}_i)]$ *for a norm which agrees with the initial norm on* M *and verifies inequality * for every polynomial.*

IV. EXAMPLES

For convenience M is a Banach space.

1) Let M_1' be the unit ball of the dual space of M endowed with the $\sigma\,(\mathrm{M}', \mathrm{M})$-topology.

M can be viewed as the space of continuous linear functionals on M_1'. By theorem 3.1 of (2) M has the m.e.p. in $\mathrm{C}\,(\mathrm{M}_1')$, the algebra of continuous functions on M_1', and the algebra generated by M in $\mathrm{C}\,(\mathrm{M}_1')$ verifies conditions i), ii) and iii).

Here $\|p\|_1 = \sup\limits_{\mathrm{L}\in \mathrm{M}_1'} |p\,[\mathrm{L}\,(x_1), \cdots, \mathrm{L}\,(x_n)]| =$ spectral radius of p.

2) Let M^n be the n-th tensor power of M and $\mathrm{M}^{\hat{i}}$ the projective n-th power $(\mathrm{M}^{\hat{}} = \mathrm{M}^{\widehat{n-1}} \widehat{\otimes}\, \mathrm{M})$. Put

$$\mathrm{T}\,(\mathrm{M}) = \left\{ x = (x_n) \in \prod_{n=0}^{\infty} \mathrm{M}^{\hat{n}} : \|x\| = \sum_{n=0}^{\infty} \|x_n\| < \infty \right\}$$

T (M) is a Banach algebra under the product

$$(x \cdot y)_n = \sum_{k+r=n} x_k \otimes y_r$$

Let K be the closed ideal generated by the set $\{x \otimes y - y \otimes x : x, y \in \mathrm{M}\}$.

Then the algebra T(M)/K contains an isometric copy of M and has properties ii) and iii).

For this construction see Leptin (1).

It should be noted that in the case $M = \mathbf{C}$ construction 1) leads to the disc algebra and construction 2) to the algebra of analytic functions on the closed unit disc with absolutely convergent Fourier series.

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