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Basis in a certain Completion of the s-d-ring over the rational Numbers

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 13 maggio 1978

Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Algebra. — *Basis in a certain Completion of the s-d-ring over the rational Numbers.* Nota I di ESAYAS GEORGE KUNDERT, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si determinano diverse basi per un completamento di un *s*-*d*-anello sopra il campo razionale.

INTRODUCTION

In [5] we promised to develop an "analysis" in the completion $\hat{\mathscr{A}}$ of the *s*-*d*-ring $\mathscr{A}$ over the integers. It turns out to be expedient, to first replace the integers by the field **Q** of rational numbers.

In the following article we make an attempt to introduce certain basis in $\hat{\mathscr{A}}$. Each element can then be represented by a series with respect to such a basis. Each basis is defined with help of an operator in $\hat{\mathscr{A}}$ and the coefficients in the series may be expressed in terms of the operator. If an operator A defines an A-basis, then we show that the dual operator $A' = E - A$ defines always an A′-basis.

We illustrate the concept with three typical examples and their duals. With help of these examples we can, at the same time, show:

(1) The classical difference calculus is in a certain sense subordinated to our analysis.

(*) Nella seduta del 13 maggio 1978.

(2) The classical Stirling numbers appear as coefficients for certain basis transformations and we have therefore a natural and new interpretation of those numbers.

(3) The classical Bernoulli numbers appear in three new interpretations.

Let $\hat{\mathscr{A}}$ be the completion of the *s-d*-ring $\mathscr{A}$ with respect to the inteal $m = (x_1)^*$ over the field of rational number $\mathbf{Q}$. (See [4–6, 2] for notations and definitions used in this paper). Let A be a linear mapping from $\hat{\mathscr{A}} \to \hat{\mathscr{A}}$.

DEFINITION. An A-basis of $\hat{\mathscr{A}}$ is a sequence $\{z_n\}$ with $z_n \in \hat{\mathscr{A}}$ such that there exists a subalgebra N_A of $\hat{\mathscr{A}}$ and a N_A-algebra homomorphism σ_A from $\hat{\mathscr{A}}$ onto N_A and such that for each element $a \in \hat{\mathscr{A}}$ we have $a = \sum_{n=0}^{\infty} (\sigma_A A^n a) z_n$ uniquely.

Note that $1 \in N_A$ and that $\sigma_A(\alpha) = \alpha$ for $\alpha \in N_A$. Since $z_n = 1 \cdot z_n$ it follows that $\sigma_A A^m z_n = 0$ if $m \neq n$ and $\sigma_A A^n z_n = 1$, especially $\sigma_A(z_n) = 0$ for $n > 0$.

Let $Az_0 = \sum_{n=0}^{\infty} \alpha_n z_n$ then $\sigma_A A^m z_0 = 0 = \alpha_{m-1}$ for $m \geq 1$. Therefore $Az_0 = 0$.

Let $Az_1 = \sum_{n=0}^{\infty} \beta_n z_n$ then $\sigma_A Az_1 = \beta_0 = 1$ and $\sigma_A A^m z_1 = \beta_{m-1} = 0$ for $m \geq 2$.

It follows that $Az_1 = z_0$. Similarly one gets $Az_n = z_{n-1}$ for $n > 1$.

Next, if $a = \sum_{n=0}^{\infty} \alpha_n z_n$ then $\alpha_n = \sigma_A A^n a$ and

$$Aa = \sum_{n=0}^{\infty} (\sigma_A A^{n+1} a) z_n = \sum_{n=0}^{\infty} (\sigma_A A^{n+1} a) Az_{n+1}$$

or

$$Aa = \sum_{n=0}^{\infty} \alpha_n Az_n .$$

Especially also $A(\alpha \cdot a) = \alpha \cdot A(a)$ for $\alpha \in N_A$ and $a \in \hat{\mathscr{A}}$, so that A is automatically N_A-linear. Since

$$A\left(\sum_{n=0}^{\infty} \alpha_n z_{n+1}\right) = \sum_{n=0}^{\infty} \alpha_n Az_{n+1} = \sum_{n=0}^{\infty} \alpha_n z_n = a,$$

it follows furthermore that A must be surjective.

Examples of linear mappings with an A-basis:

Example (1). Let $A = D$, $N_D = \operatorname{Ker} D = \mathbf{Q}$, $\sigma_D = \sigma$ then $\{x_n\}$ (see [4] for definition of σ and x_n) is a D-basis. In this example we could also take $\mathbf{Z}$ as ground ring in place of $\mathbf{Q}$.

Example (2). Let $A = D_2 = (K^{-2})' = E - K^{-2} = (2 - D) D$. (See [5] for definition of K) and take $N_{D_2} = \operatorname{Ker} D_2 = \{\alpha + \beta e\}$ where $e = \sum_{n=1}^{\infty} 2^{n-1} x_n$, $\alpha, \beta \in \mathbf{Q}$.

Note that $e^2 = -e$. Let $c \in N_{D_2}$ and $a \in \hat{\mathscr{A}}$, then recalling that D_2 is a semi-derivation, we have $D_2(c \cdot a) = c \cdot D_2 a + 0 \cdot a - 0 \cdot D_2 a$. It follows that D_2 is N_{D_2}-linear. One checks easily that D_2 is surjective, for this it is important that the ground ring is **Q** and not **Z**. Now let $\sigma_{D_2}(x_1) = e$ which defines σ_{D_2} uniquely (see Formula (I), [5]), as a matter of fact, it follows that $\sigma_{D_2}(x_n) = 0$ for $n \geq 2$ and therefore if $a = \sum_{n=0}^{\infty} \alpha_n x_n$, then $\sigma_{D_2}(a) = \alpha_0 + \alpha_1 e \in N_{D_2}$. Especially $\sigma_{D_2}(c) = c$ if $c \in N_{D_2}$.

Let $S_{D_2}(a) = a' - \sigma_{D_2}(a')$, where $a' \in \hat{\mathscr{A}}$ such that $D_2(a') = a$, then S_{D_2} has the following properties:

(*a*) S_{D_2} is well-defined [since if a'' is another element such that $D_2(a'') = a$ then $D_2(a' - a'') = a - a = 0 \Rightarrow a' - a'' \in N_{D_2} \Rightarrow \sigma_{D_2}(a' - a'') = a' - a''$ or $a' - \sigma_{D_2}(a') = a'' - \sigma_{D_2}(a'')$].

(*b*) S_{D_2} is N_{D_2}-linear [since $S_{D_2}(\alpha \cdot a) = (\alpha \cdot a)' - \sigma_{D_2}(\alpha \cdot a)' = \alpha \cdot a' - \alpha \cdot \sigma_{D_2}(a') = \alpha \cdot S_{D_2}(a)$].

(*c*) $D_2 S_{D_2} = E$

(*d*) $S_{D_2} D_2 = E - \sigma_{D_2} = \sigma'_{D_2}$

(*e*) $\sigma_{D_2} S_{D_2} = 0$.

Let now $y_n = S_{D_2}^n(1) = (-1)^n \sum_{k=2n}^{\infty} \binom{k-n-1}{n-1} 2^{k-2n} x_k$. It is clear from the above properties that $D_2 y_n = y_{n-1}$ and $\sigma_{D_2}(y_n) = 0$ for $n \geq 1$.

We assert that $\{y_n\}$ is a D_2-basis. To prove this, let $a = \sum_{m=0}^{\infty} \alpha_m x_m$ be an arbitrary element of $\hat{\mathscr{A}}$. Now if $\{y_n\}$ is a D_2-basis, we should have

$$a = \sum_{n=0}^{\infty} (\sigma_{D_2} D_2^n a)\, y_n\,, \qquad \text{but} \quad \sigma_{D_2} D_2^n a = \sum_{m=0}^{\infty} \alpha_m \sigma_{D_2} D_2^n x_m\,,$$

$$\text{so} \quad a = \sum_{n=0}^{\infty} \left[\left(\sum_{k=n+1}^{2n+1} (-1)^{n+1+k} \binom{n}{k-n-1} 2^{2n-k+1} \alpha_k \right) e + \left(\sum_{k=n}^{2n} (-1)^{n+k} \binom{n}{k-n} 2^{2n-k} \alpha_k \right) \right] y_n\,.$$

It is sufficient to check this for $a = x_m$ by substituting the series expression given above for y_n. Note that we could have taken $A = D_n$ (See [6]) to obtain infinitely many examples all similar to example (1) and (2). Next we will give an example where the operator A is not a semi-derivation.

Example (3). Let $A = H = E - DQ_1$ where Q_1 is the operator defined by $Q_1(a) = x_1 \cdot a$. Let $N_H = \operatorname{Ker} H = \mathbf{Q}$ and $\sigma_H = \sigma$ as in example (1). One checks easily that H is onto, but for this it is again important that the ground ring is **Q** and not **Z**. Let $u_n = S_H^n(1)$, where S_H is defined as in example (2) replacing D_2 by H. Computing this it turns out that $u_n = \sum_{k=n}^{\infty} (-1)^k C_n^k x_k$, where the C_n^k are the Stirling numbers of the 1. kind. (See [3] for the definition used here). Computing $\sigma H^n x_k$ one gets for the H-series of x_k the series $(-1)^k \sum_{n=0}^{\infty} B_n^k u_n$, where the B_n^k are the Stirling numbers of the 2. kind. Substituting the above series for u_n and using simple properties of Stirling numbers, one sees that this series is indeed $= x_k$.

This in terms guarantees that $\{u_n\}$ is a H-basis.

Let A be any operator with an A-basis $\{z_n\}$ and let $A' = E - A$, then we can always construct an A′-basis $\{z'_n\}$ as follows: Put $N_{A'} = N_A$, $\sigma_{A'} = \sigma_A$ and $z'_n = (-1)^n \sum_{k=n}^{\infty} \binom{k}{n} z_k$. Note that in this case z'_0 is not equal to one. We have at once $\sigma_{A'}(z'_0) = 1$ and $\sigma_A(z'_n) = 0$ for $n \geq 1$.

Also:

$$A' z'_n = (-1)^n \sum_{k=n}^{\infty} \binom{k}{n} (z_k - z_{k-1}) = (-1)^{n-1} \sum_{k=n-1}^{\infty} \binom{k}{n-1} z_k = z'_{n-1}.$$

Furthermore

$$z_n = (-1)^n \sum_{k=n}^{\infty} \binom{k}{n} z'_k$$

and since

$$x_m = \sum_{n=0}^{\infty} \beta_{mn} z_n \qquad \text{with} \quad \beta_{mn} \in N_{A'} \Rightarrow$$

$$x_m = \sum_{n=0}^{\infty} \beta_{mn} (-1)^n \sum_{k=n}^{\infty} \binom{k}{n} z'_k = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} (-1)^n \binom{k}{n} \beta_{mn} \right) z'_k = \sum_{k=0}^{\infty} \gamma_{mk} z'_k$$

with $\gamma_{mk} \in N_{A'}$ which shows that $\{z'_n\}$ is an A′-basis.

Example (1′). Let $A = D$ and therefore $A' = E - D = K^{-1}$ (See [5]). Since K^{-1} is a **Q**-homomorphism from $\hat{\mathscr{A}}$ onto $\hat{\mathscr{A}}$ with

$$\text{kernel} = \{\alpha \cdot x'_0\}, \qquad \text{so} \quad a = \sum_{m=0}^{\infty} \alpha_m x_m = \sum_{m=0}^{\infty} \alpha_m \sum_{k=0}^{\infty} \gamma_{mk} x'_k =$$

$$= \sum_{k=0}^{\infty} \left(\sum_{k=0}^{\infty} \alpha_m \gamma_{mk} \right) x'_k = \sum_{k=0}^{\infty} \alpha'_k x'_k \Rightarrow \alpha'_k = \sigma K^{-k} a.$$

Now if $b = \sum_{k=0}^{\infty} \beta'_k x'_k$ with $\beta'_k = \sigma K^{-k} b$,

we have

$$a \cdot b = \sum_{k=0}^{\infty} \sigma K^{-k} (ab) x'_k = \sum_{k=0}^{\infty} (\sigma K^{-k} a)(\sigma K^{-k} b) x'_k = \sum_{k=0}^{\infty} \alpha'_k \beta'_k x'_k$$

because σK^{-k} is a homomorphism.

Let $\hat{A}_1$ be the **Q**-algebra of sequences (α_k) where $\alpha_k \in \mathbf{Q}$ and define $d_1(\alpha_k) = (\alpha_k - \alpha_{k+1})$. The mapping d_1 is a semi-derivation in $\hat{A}_1$. We may then turn $\hat{A}_1$ into a *s-d*-ring. (See [2, 4]). Let Δ_1 be the mapping $\hat{\mathscr{A}} \to \hat{A}_1$ which associates to $a = \sum_{k=0}^{\infty} \alpha_k x_k'$ the sequence (α_k). It is clear that Δ_1 is surjective and injective. From the above it follows that Δ_1 is an algebra-isomorphism.

Since

$$x_n = (-1)^n \sum_{k=0}^{\infty} \binom{k}{n} x_k'$$

and

$$Dx_n = x_{n-1} = (-1)^{n-1} \sum_{k=0}^{\infty} \binom{k}{n-1} x_k' = (-1)^{n-1} \sum_{k=0}^{\infty} \left[\binom{k+1}{n} - \binom{k}{n}\right] x_k'$$

so that $\Delta_1 Dx_n = d_1 \Delta_1 x_n$ and from this follows that Δ_1 preserves also semiderivations. Part of the structure of $\hat{\mathscr{A}}$ is therefore (algebraically) isomorphic to $\hat{A}_1$ and we have a new way to investigate the classical difference calculus. By using the topology on $\hat{\mathscr{A}}$, we can now use—formally—the methods of modern analysis. Furthermore difference calculus appears now in an axiomatic setting.

Vice versa we can get results for $\hat{\mathscr{A}}$ from known facts in difference calculus.

For example, if we would like to know what the series expansion of x_n^m with respect to the basis $\{x_j\}$ is, we can argue as follows:

Since $\Delta_1 (x_n) = \left[(-1)^n \binom{k}{n}\right]$ it follows that

$$\Delta_1 (x_n^m) = \left[(-1)^{nm} \binom{k}{n}^m\right]$$

so $$x_n^m = (-1)^{nm} \sum_{k=0}^{\infty} \binom{k}{n}^m x_k' = (-1)^{nm} \sum_{k=0}^{\infty} \binom{k}{n}^m (-1)^k \sum_{j=0}^{\infty} \binom{j}{k} x_j =$$

$$= (-1)^{nm} \sum_{j=n}^{\infty} \left[\sum_{k=n}^{\infty} (-1)^k \binom{j}{k} \binom{k}{n}^m\right] x_j .$$

For $n = 1$ we get:

$$x_1^m = \sum_{j=1}^{\infty} \left[(-1)^m \sum_{k=1}^{j} (-1)^k \binom{j}{k} k^m\right] x_j \Rightarrow x_1^m = \sum_{j=1}^{\infty} (-1)^{m+j} B_j^m x_j$$

where the B_j^m are the Stirling numbers of the second kind.

One might ask whether $\{x_1^m\}$ is a A-basis for some operator A in $\hat{\mathscr{A}}$? It is clear that this would imply that $\lim_{m\to\infty} x_1^m = 0$, which is, however, not the case. For the subalgebra $\mathscr{A}$, however, $\{x_1^m\}$ is a A-basis, namely for the operator A defined by $Ax_n = \frac{1}{n} \sum_{k=0}^{n} x_k$. For this basis we have $x_j = \sum_{m=0}^{j} (-1)^{j+m} C_m^j x_1^m$ where C_m^j are the Stirling numbers of the first kind.

Taking $\{x_n^m\}$ as a basis of $\mathscr{A}$ then the numbers

$$_n B_j^m = (-1)^j \sum_{k=n}^{j} (-1)^k \binom{j}{k} \binom{k}{n}^m$$

with fixed n would appear to be a natural generalization of the Stirling numbers of the second kind and the elements of the inverse matrix $_n C_m^j$ a generalization of the Stirling numbers of the first kind.

LITERATURE

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