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Boundedness of solutions of an n-th order nonlinear differential equation

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Equazioni differenziali ordinarie. — *Boundedness of solutions of an n-th order nonlinear differential equation* (*). Nota di STANISŁAW SĘDZIWY, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si danno condizioni sufficienti per la limitatezza globale delle soluzioni di un'equazione differenziale ordinaria non lineare di ordine n.

1. Consider the n-th order differential equation

$$y^{(n)} + g_1(y^{(n-2)})\, y^{(n-1)} + \cdots + g_{n-1}(y)\, y' + f(y\,;y',\cdots,y^{(n-1)}) = p(t)\,, \tag{1.1}$$

where the functions $g_i : R \to R$, $f : R \times R^{n-1} \to R$, $p : R \to R$ are continuous (R^k denotes the real Euclidean k-space with the norm $|x|$).

The Note presents sufficient conditions for the global boundedness of solutions of (1.1), which generalize the results of [1] and [6]. The method of proof is based on the theory developed in [6]. It also permits to establish the existence of an ω-periodic solution of (1.1) when p is periodic with period ω.

Recall that solutions of (1.1) are said to be globally bounded if there exists an $e > 0$ such that any solution $y = y(t)$ of (1.1) satisfies $\sum_{i=0}^{n-1} |y^{(i)}(t)| < e$ for $t \geq T$, where T depends only on $y(t)$.

For an $m \times k$ matrix A, A^T denotes the matrix transposed to A. I is the unit matrix. If A is a square matrix, A^{-1} denotes the matrix inverse to A. A is said to be stable if its eigenvalues have negative real parts.

THEOREM. *Let* $G_i(u) = \int_0^u g_i(s)\, ds$ *satisfy*

$$|G_i(u) - a_i u| \leq \mu |u| \qquad \text{for} \quad i = 1, 2, \cdots, n-1\,, \ |u| \geq u_1\,. \tag{1.2}$$

Let the polynomial $\varphi(\lambda) = \lambda^{n-1} + a_1 \lambda^{n-2} + \cdots + a_{n-1}$ *have roots with negative real parts. Let*

$$0 < yf(y\,;z) \leq \mu y^2 \qquad \text{for} \quad |y| \geq r_0\,, \qquad z \in R^{n-1}\,. \tag{1.3}$$

(*) This Note was written while the Author was Visiting the University of Utah, Autumn 1973.

(**) Nella seduta dell'8 aprile 1978.

Assume that $p(t)$ *satisfies one of the conditions*

$$|P(t)| = \left| \int_0^t p(s)\, ds \right| \leq m_1 \quad \textit{for} \quad t \in R, \tag{1.4}$$

$$|p(t)| \leq m_2 < m_3 \leq |f(y; z)| \quad \textit{for} \quad t \in R, \quad |y| \geq r_1, \quad z \in R^{n-1}. \tag{1.5}$$

Then for all sufficiently small μ *the solutions of* (1.1) *are globally bounded.*

2. *Proof*. Put $z_1 = y$, $z^T = (z_1, \cdots, z_{n-1})$, and $G(z) = (a_{n-1} z_1 - G_{n-1}(z_1)) + \cdots + (a_1 z_{n-1} - G_1(z_{n-1}))$.

Then (1.1) is equivalent to the system

$$\begin{aligned} z' &= Az + bz_n + bF_1(t, z) \\ z'_n &= -f(c^T z; u) + p_2(t) \\ u &= Az + bz_n + bF_1(t, z), \end{aligned} \tag{2.1}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdot & \cdots & 0 & 1 \\ -a_{n-1} & \cdot & \cdot & \cdots & -a_2 & -a_1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ 0 \\ 0 \end{bmatrix},$$

$F_1(t, z) = G(z) + p_1(t)$, and $p_1(t) = P(t)$, $p_2(t) \equiv 0$ if (1.4) holds, or $p_1(t) \equiv 0$, $p_2(t) = p(t)$ otherwise.

Note that in both cases p_1 and p_2 are bounded. The matrix A is stable. Since $c^T A^{-1} b = 1/(-a_{n-1}) \neq 0$, the transformation

$$x = Az + bz_n, \qquad s = c^T z \tag{2.2}$$

is nonsingular. Applying to (2.1) the change of variables (2.2) and using the formula $c^T b = 0$, we get the system

$$\begin{aligned} x' &= Ax - b(f(s; w) - p_2(t)) + Abg(t, x, s) \\ s' &= c^T x \\ w &= x + bg(t, x, s), \end{aligned} \tag{2.3}$$

where we define $g(t, Az + bz_n, c^T z) \equiv F_1(t, z)$.

By (1.2), $|F_1(t, z)| \leq \mu(n-1)|z| + m$ for arbitrary $z \in R^{n-1}$, $t \in R$, hence

$$|g(t, x, s)| \leq \mu k(|x| + |s|) + m, \quad x \in R^{n-1}, \quad s \in R, \quad t \in R, \tag{2.4}$$

for certain $k > 0$, $m > 0$.

Similarly, from (1.3) and (1.5) it follows that

$$(2.5) \qquad 0 < s(f(s; w) - p_2(t)) < \mu s^2 + m_2 s \quad \text{if} \quad |s| \geq r_2, \quad 0 < \mu \leq \mu_1,$$

where μ_1 is a suitable constant.

Put $V(x) = x^T Lx$, $W(x, s) = q^T x + \gamma s$, where $q = -a_{n-1}(A^T)^{-1} c$, $\gamma = a_{n-1}$ and L is the $n \times n$ symmetric matrix satisfying

$$(2.6) \qquad A^T L + LA = -I.$$

L is uniquely defined by (2.6) and positive definite (cf. [2, p. 189]). Let l_1^2, l_2^2 be respectively the smallest and largest eigenvalue of L.

Now let $D(\alpha)$ $(\alpha > 0)$ be the family of sets

$$D(\alpha) = \{(x, s) : V(x) \leq \alpha^2, |W(x, s)| \leq \beta(\alpha)\},$$

where $\beta(\alpha) = |q|(\alpha + l)/l_1$, and $|q| l \geq \gamma r_2 l_1$. The boundary of $D(\alpha)$ is the union of sets $D_1(\alpha) = \{(x, s) : V(x) = \alpha^2, |W(x, s)| \leq \beta(\alpha)\}$, $D_2(\alpha) = \{(x, s) : V(x) \leq \alpha^2, |W(x, s)| = \beta(\alpha)\}$. From the formulas $c^T b = 0$, $A^T q + \gamma c = 0$, and $q^T b = 1$ it follows that the derivative of W relative to (2.3) is $W'(x, s) = -(f(s; w) - p_2(t))$.

If $(x, s) \in D_2(\alpha)$, then $\gamma |s| \geq l|q|/l_1 \geq \gamma r_2$, hence by (2.5),

$$(2.7) \qquad W'(x, s) \operatorname{sgn} s < 0 \quad \text{for} \quad (x, s) \in D_2(\alpha), \alpha > 0.$$

From (2.6) it follows that $V'(x) = -|x|^2 - 2x^T Lb(f(s; w) - p_2(t)) + 2x^T LAbg(t, x, s)$. So (2.4) and (2.5) imply that

$$(2.8) \qquad V'(x) \leq -|x|^2 + 2|Lb|\mu|s||x| + 2|LAb|\mu k|x|(|x| + |s|) + 2|LAb|m|x| + 2|Lb|m_2|x|.$$

Let $(x, s) \in D_1(\alpha)$. Then $\alpha/l_2 \leq |x|$, $|s| \leq |q|(2l_2|x| + 1)/(\gamma l_1)$ and, by (2.8),

$$V'(x) \leq -|x|^2 + k_1(\mu)|x|^2 + k_2(\mu)|x|.$$

where $k_1(\mu)$, $k_2(\mu)$ are in terms of L and the coefficients of (2.3). Moreover $\lim_{\mu \to 0} k_1(\mu) = 0$. Hence if μ_1 is so small that $k_1(\mu_1) < 1$, then there is an $\alpha_0 > 0$ such that

$$(2.9) \qquad V'(x) < 0 \quad \text{for} \quad (x, s) \in D_1(\alpha), \alpha > \alpha_0.$$

(2.7) and (2.9) imply that any solution of (2.3) entering $D(\alpha)$, $\alpha \geq \alpha_0$ at $t = t_0$ remains in $D(\alpha)$ for all $t > t_0$.

Since $\lim_{|x| \to \infty} V(x) = \infty$ and $\lim_{|s| \to \infty} |W(x, s)| = \infty$ uniformly for x in an arbitrary compact set, $D(\alpha)$ is compact for $\alpha > 0$. Further, for any (x, s) there is an α such that $(x, s) \in D(\alpha)$.

A compactness argument shows that for any $\alpha > \alpha_0$, there is an $m(\alpha) < 0$ such that $V' < m(\alpha)$, $W' < m(\alpha)$ in $D(\alpha) \setminus D(\alpha_0)$. So every solution eventually enters $D(\alpha_0)$.

Since $D(\alpha_0)$ is bounded, from the above it follows that the solutions of (2.3) are ultimately bounded. This ends the proof of the theorem.

3. COROLLARY (see [5]). *If the assumptions of the theorem hold and $p(t)$ is periodic with period ω, then* (1.1) *has at least one ω-periodic solution.*

For the proof, in the case of uniqueness of the initial value problem, see [4] or [6]. Since (2.3) may be approximated in $D(\alpha_0)$ by equations having the property of uniqueness and such that the corresponding periodic solutions are in $D(\alpha_0)$, by a standard limiting argument (see [3, Theorem 2.4]) we conclude that the corollary is also true in the general case.

Note that the corollary also holds if the assumption on φ is relaxed to the following: the roots of φ are different from $2\pi ip/\omega$ $p = 1, 2, \cdots$ (see [7, Theorem 3]).

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