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A note on Boundary Value Problems at Resonance

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Equazioni differenziali ordinarie. — *A note on Boundary Value Problems at Resonance.* Nota di MARIO MARTELLI, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra che l'equazione differenziale

$$x'' + f(x)\,x' + g(t, x) = e(t)$$

con $f(x)$ continua, $g(t, x)$ continua e tale che $g(t, x) = g(t+1, x)$, $e(t) = e(t+1)$ possiede una soluzione periodica di periodo 1 anche nel caso in cui la funzione

$$G(t) := \liminf_{|x| \to +\infty} \frac{g(t, x)}{x}$$

oltrepassa il primo autovalore (0) del problema lineare associato in un insieme di misura non nulla purché $G(t)$ sia ad integrale positivo e il

$$\limsup_{|x| \to +\infty} \frac{|g(t, x)|}{|x|}$$

non si avvicini al secondo autovalore ($4\pi^2$).

In a recent paper J. Mawhin [5] proved that the problem

$$(1) \qquad \begin{cases} x'' + g(t, x) = 0 \\ x(0) = x(1) \\ x'(0) = x'(1) \end{cases}$$

where f satisfies Caratheodory conditions, has a solution provided that

$$\text{(i)} \qquad F(t) = \limsup_{|x| \to +\infty} \frac{g(t, x)}{x} \leq 0$$

uniformly with respect to $t \in [0, 1]$ and

$$\text{(ii)} \qquad \int_0^1 F(t)\, dt < 0 .$$

In the above paper J. Mawhin observed that $F(t)$ can be equal to the first eigenvalue (0) of the associated linear problem

$$(2) \qquad \begin{cases} x'' + \mu x = 0 \\ x(0) = x(1) \\ x'(0) = x'(1) \end{cases}$$

(*) Nella seduta dell'8 aprile 1978.

in a set of positive measure provided that condition (ii) holds. He also posed the question on whether or not $F(t)$ can overtake the first eigenvalue, i.e. whether or not (i) can be replaced by the more general assumption

$$\text{(i')} \qquad F(t) \leq a$$

for some positive constant a.

The purpose of the present note is to show that the first eigenvalue of the associated linear problem (2) can be overtaken from above in a set of positive measure provided that a condition similar to (ii) is satisfied by the function

$$G(t) = \liminf_{|x| \to +\infty} \frac{g(t, x)}{x}$$

namely

$$\int_0^1 G(t) \, dt > 0$$

while $F(t)$ should be bounded away from the second eigenvalue ($4\pi^2$). This way we extend many previous results (A. C. Lazer [3], J. Mawhin [5], M. Martelli [4], S. H. Chang [1]), where $g(t, x)/x$, which was independent of t in the papers of Lazer and Mawhin, was not allowed to overtake the first eigenvalue and it was bounded away from the second eigenvalue for $|x|$ big enough.

It is known that further existence theorems for problem (1) have been obtained in the case when $g(x)/x$ (this time g is independent of t) is contained in an interval boundend away from two consecutive eigenvalues when $|x|$ is big enough (see R. Reissig [6]). We do not know if our result can be extended to this situation via a suitable modification of the assumptions. It should be pointed out, however, that while the first eigenvalue has geometric multiplicity 1, the other ones have geometric multiplicity 2. This fact is maybe the reason why the extension needs more work, if it is possible.

In proving our theorem we will use the following result (see for example, M. Furi, M. Martelli e A. Vignoli [1], Theorem 5.2.1).

THEOREM 1. *Let* $A : E \to F$ *be a linear isomorphism acting between the Banach spaces* E *and* F *and let* $h : E \times [0, 1] \to F$ *be compact and such that* $h(x, 0) = 0$ *for every* $x \in E$. *Assume that*

$$S = \{x \in E : Ax = h(x, \lambda) \quad \textit{for some} \quad \lambda \in [0, 1]\}$$

is bounded. Then the equation

$$Ax = h(x, 1)$$

has a solution $x \in E$.

Let the boundary value problem

$$(3)\qquad \begin{cases} x'' + f(x)\,x' + g(t\,,x) = e(t) \equiv e(t+1) & t \in \mathbf{R} \\ x(0) = x(1) \\ x'(0) = x'(1) \end{cases}$$

be given, where $f(x)$ is continuous and $g(t\,,x)$ is continuous and periodic of period 1 with respect to t.

We have

THEOREM 2. *The boundary value problem* (3) *has a solution provided that*

$$\text{(i)}\qquad \limsup_{|x|\to+\infty} \left| \frac{g(t\,,x)}{x} \right| < \frac{4\pi^2}{2\pi+1}$$

$$\text{(ii)}\qquad \int_0^1 G(t)\,dt > 0\,, \qquad \textit{where} \quad G(t) = \liminf_{|x|\to+\infty} \frac{g(t\,,x)}{x}\,,$$

uniformly with respect to t.

Proof. Let the boundary value problem be given

$$(4)\qquad \begin{cases} x'' + \mu x + \lambda\,[f(x)\,x' + g(t\,,x) - e(t)] = 0 & \mu > 0 \\ x(0) = x(1) \\ x'(0) = x'(1)\,. \end{cases}$$

On the basis of Theorem 1 it is easy to see that to obtain the existence of a solution of problem (3) it is enough to prove that when λ ranges between 0 and 1 the set of solutions of problem (4) is bounded in $C^2[0\,,1]$, independently of μ. In fact, in this case, letting $\mu \to 0$ and applying Ascoli–Arzela's theorem we obtain a solution for problem (3).

The first step of our proof is to show that ther exists a constant M, independent of $\mu > 0$, such that for any solution $x(t)$ of (4) there exists a point $t_0 \in [0\,,1]$ such that

$$|x(t_0)| \le M\,.$$

Let N and ε_0 be such that

$$\varepsilon_0 + \frac{1}{N} \int_0^1 |e(t)|\,dt < \int_0^1 G(t)\,dt\,.$$

There exists M_{ε_0} such that whenever $|x| \ge M_{\varepsilon_0}$ we have

$$G(t) - \varepsilon_0 < \frac{g(t\,,x)}{x} \qquad \text{for every} \quad t \in [0\,,1]\,.$$

Let $M = \max\{N, M_{\varepsilon_0}\}$. Assume that a solution of (4) is such that

$$|x(t)| \geq M$$

i.e. either $x(t) \geq M$ or $x(t) \leq -M$. We have

$$\frac{x''}{x} + \mu + \lambda \frac{f(x)\, x'}{x} + \lambda \frac{g(t, x)}{x} = \lambda \frac{e(t)}{x}. \tag{5}$$

Integrating (5) we obtain

$$\int_0^1 \frac{x'^2}{x^2} + \mu + \lambda \int_0^1 \frac{g(t, x)}{x} = \lambda \int_0^1 \frac{e(t)}{x}. \tag{6}$$

Since $\frac{g(t, x)}{x} > G(t) - \varepsilon_0$ we obtain

$$\int_0^1 \frac{x'^2}{x^2} + \mu + \lambda \int_0^1 G(t) \leq \lambda \int_0^1 \frac{e(t)}{x} + \lambda\varepsilon_0. \tag{7}$$

Therefore

$$\int_0^1 \frac{x'^2}{x^2} + \mu + \lambda \int_0^1 G(t) \leq \lambda\varepsilon_0 + \frac{\lambda}{M} \int_0^1 |e(t)|. \tag{8}$$

This contradiction shows that $|x(t_0)| \leq M$ for some $t_0 \in [0, 1]$.

Let us write

$$x(t) = a + u(t)$$

where

$$a = \int_0^1 x(t)\, dt.$$

From $x(t_0) = a + u(t_0)$ we obtain that

$$|a| \leq M + |u(t_0)|.$$

On the other hand

$$|u(t)| \leq \int_0^1 |u'(t)|\, dt.$$

Therefore, by Hölder's inequality

$$|a| \leq M + \left(\int_0^1 u'^2\, dt\right)^{1/2}.$$

From (4) we obtain

$$x'' x + \mu x^2 + \lambda f(x) x' x + \lambda g(t, x) x = \lambda e(t) x \tag{9}$$

which, after integration, leads to

$$-\int_0^1 x'^2 + \mu \int_0^1 xa + \mu \int_0^1 xu + \lambda \int_0^1 g(t, x) a + \lambda \int_0^1 g(t, x) u = \lambda \int_0^1 e(t) a + \lambda \int_0^1 e(t) u . \tag{10}$$

Since $x(t)$ is a solution of (4) we see that

$$\mu \int_0^1 xa + \lambda \int_0^1 g(t, x) a = \lambda \int_0^1 e(t) a .$$

Hence (10) reduces to

$$\int_0^1 x'^2 = \mu \int_0^1 u^2 + \lambda \int_0^1 g(t, x) u - \lambda \int_0^1 e(t) u . \tag{11}$$

Therefore

$$\int_0^1 x'^2 \leq \mu \int_0^1 u^2 + \int_0^1 | g(t, x) | \, | u | + \int_0^1 | e(t) | \, | u | . \tag{12}$$

Since $\limsup_{|x| \to +\infty} \frac{| g(t, x) |}{| x |} < \frac{4 \pi^2}{2 \pi + 1}$ there exist two positive constants A, B with

$$B < \frac{4 \pi^2}{2 \pi + 1}$$

such that

$$| g(t, x) | \leq A + B | x | .$$

From (12) we obtain now, with $E = \max | e(t) |$,

$$\int_0^1 x'^2 \leq (A + E + B | a |) \int_0^1 | u | \, dt + (B + \mu) \int_0^1 u^2 \, dt . \tag{13}$$

Using Wirtinger's inequality we have

$$\int_0^1 u'^2 \, dt \leq \frac{A + E + B\,|a|}{2\pi} \left(\int_0^1 u'^2 \, dt\right)^{1/2} + \frac{B + \mu}{4\pi^2} \left(\int_0^1 u'^2 \, dt\right).$$

This implies that

$$\left(\int_0^1 u'^2 \, dt\right)^{1/2} \leq T + r\,|a| \tag{14}$$

with $T = \dfrac{2\pi(A + E)}{4\pi^2 - B - \mu}$ and $r = \dfrac{2\pi B}{4\pi^2 - B - \mu}$ which is less that 1 if μ is small enough.

From

$$|a| \leq M + \left(\int_0^1 u'^2 \, dt\right)^{1/2}$$

we now obtain that

$$|a| \leq M + T + r\,|a|$$

which implies

$$|a| \leq \frac{M + T}{1 - r}.$$

This inequality placed in (14) gives

$$\left(\int_0^1 u'^2 \, dt\right)^{1/2} \leq T + \frac{r}{1 - r}(M + T) = \frac{T + rM}{1 - r}.$$

Hence

$$\|x\| \leq \frac{M + T}{1 - r} + \frac{T + rM}{1 - r} = \frac{2T + (1 + r)M}{1 - r}. \tag{15}$$

Now the bounds on $\|x'\|$ and $\|x''\|$ can be obtained in a straightforward manner.

Remark. The condition

$$\limsup_{|x| \to +\infty} \frac{|g(t, x)|}{|x|} < \frac{4\pi^2}{2\pi + 1}$$

uniformly with respect to t, can be replaced by the more general assumption

$$| g(t, x) a \leq h(t) + B | x | \qquad t \in [0, 1], x \in \mathbf{R}$$

with $B < \frac{4\pi^2}{2\pi + 1}$ and $\int_0^1 h^2(t)\, dt < +\infty$.

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