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Remarks on the oscillation theorem for a nonlinear even order damped differential equations

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Equazioni differenziali ordinarie. — *Remarks on the oscillation theorem for a nonlinear even order damped differential equations.* Nota di LU-SAN CHEN, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si danno condizioni sulle funzioni p, q, f, g e h sotto le quali tutte le soluzioni continuabili di

$$u^{(n)}(t) + q(t)\,h(u(t),\cdots,u^{(n-1)}(t)) + p(t)\,(u(t))^{\alpha}\,g(u'(t),\cdots,u^{(n-1)}(t)) = 0,$$

sono oscillatorie su $[0, \infty)$ quando $\alpha > 0$, $\alpha \neq 1$ è il quoziente di due interi dispari e n è un intero pari. Se $n = 2$ questi risultati si riducono ai ben noti teoremi sufficienti di Baker, e per: $n = 2$, $q(t) = 0$, $g(u'(t),\cdots,u^{(n-1)}(t)) = 1$ si riducono anche ai teoremi sufficienti di Atkinson e Belohorec.

1. INTRODUCTION

In this paper we are concerned with the oscillatory behavior of the solutions of the nonlinear even order damped differential equation

$$u^{(n)}(t) + q(t)\,h(u(t),\cdots,u^{(n-1)}(t)) + p(t)\,(u(t))^{\alpha}\,g(u'(t),\cdots,u^{(n-1)}(t)) = 0. \tag{1}$$

Our main purpose is to extend to equation (1) some of the recent results of Baker [1].

A non-trivial solution $u(t)$ of (1) which exists on the interval $[0, \infty)$ will be called *oscillatory* on $[0, \infty)$ if and only if for each $t_0 \in [0, \infty)$ there is $t_1 > t_0$ such that $u(t_1) = 0$; otherwise, $u(t)$ will be called *nonoscillatory* on $[0, \infty)$. Equation (1) will be called oscillatory on $[0, \infty)$ if and only if every solution $u(t)$ of (1) is continuable and oscillatory on $[0, \infty)$.

(*) Nella seduta dell'8 aprile 1978.

Throughout this paper the following assumptions are assumed to hold:

(i) $h \in C[R^n, R]$ *and there is a constant* $M > 0$ *such that* $0 < \frac{h(x_1, \cdots, x_n)}{x_n} \leq M$ *for any* $x_n \neq 0$, *where* R *is the set of all real numbers.*

(ii) $q \in C[R^+, R]$, $q(t) \geq 0$ *for all* $t \in [0, \infty)$ *and for fixed* $T > 0$,

$$\lim_{t\to\infty} Q(t, T) = \lim_{t\to\infty} \int_T^t \exp\left\{-M \int_T^s q(z)\,dz\right\} ds = \infty .$$

(iii) $p \in C[R^+, R]$, $p(t) > 0$ *for all* $t \in [0, \infty)$.

(iv) $g \in C[R^{n-1}, R]$, $g(x_1, \cdots, x_{n-1}) \geq k > 0$ *for all* $x_1, \cdots, x_{n-1} \in R$.

LEMMA 1. *Suppose that assumptions* (i)-(iv) *hold. If* $u(t)$ *is a continuable nonoscillatory solution of*

$$u^{(n)}(t) + q(t)\, h(u(t), \cdots, u^{(n-1)}(t)) + p(t) f(u(t))\, g(u'(t), \cdots, u^{(n-1)}(t)) = 0 \tag{2}$$

on $[0, \infty)$ *then there is a* $T > 0$ *such that*

$$u(t)\, u^{(n-1)}(t) > 0$$

for $t \geq T$, *where* n *is an even integer,* $f \in C[R, R]$, $xf(x) > 0$ *for all* $x \neq 0$ *and* f *is monotonic.*

Proof. Let $u(t)$ be a continuable nonoscillatory solution of (2) on $[0, \infty)$. We may assume that $u(t) > 0$ on $[T, \infty)$, $T \geq 0$. Suppose that there is $t_0 \in [T, \infty)$ such that $u^{(n-1)}(t_0) < 0$. Then there exists $t \in [t_0, \infty)$ satisfying $u^{(n-1)}(t) < 0$. Multiplying (2) by $u^{(n-1)}(t)$ and observing that

$$\int_{t_0}^t p(s)\, f(u(s))\, g(u'(s), \cdots, u^{(n-1)}(s))\, u^{(n-1)}(s)\, ds < 0 ,$$

we find

$$\int_{t_0}^t u^{(n-1)}(s)\, u^{(n)}(s)\, ds + \int_{t_0}^t q(s)\, h(u(s), \cdots, u^{(n-1)}(s))\, u^{(n-1)}(s)\, ds > 0 , \tag{3}$$

from (3) using (i) we get the following estimate

$$(u^{(n-1)}(t))^2 > (u^{(n-1)}(t_0))^2 - 2M \int_{t_0}^t q(s)\, (u^{(n-1)}(s))^2\, ds .$$

Hence by the Langenhop inequality [6],

$$(u^{(n-1)}(t))^2 \geq (u^{(n-1)}(t_0))^2 \exp\left\{-2M \int_{t_0}^t q(s)\, ds\right\} . \tag{4}$$

Since $u^{(n-1)}(t) < 0$ on $[t_0, \infty)$, we have from (4),

$$u^{(n-1)}(t) \leq u^{(n-1)}(t_0) \exp\left\{-M \int_{t_0}^{t} q(s)\, ds\right\}. \tag{5}$$

Integrating (5) from t_0 to t, and noting (ii), we obtain

$$u^{(n-2)}(t) \leq u^{(n-2)}(t_0) + u^{(n-1)}(t_0)\, Q(t, t_0) \to -\infty, \qquad (t \to \infty),$$

which implies $\lim_{t\to\infty} u(t) = -\infty$, a contradiction. Thus $u^{(n-1)}(t) \geq 0$ on $[t_0, \infty)$. Suppose that there is $t_* \in [t_0, \infty)$ such that $u^{(n-1)}(t_*) = 0$, then (2) implies that

$$u^{(n)}(t_*) = -p(t_*) f(u(t_*))\, g(u'(t_*), \cdots, u^{(n-1)}(t_*), 0) < 0.$$

It follows that $u^{(n-1)}(t)$ cannot have a zero larger than t_0. Hence $u^{(n-1)}(t)$ is eventually of constant sign. This completes the proof of the Lemma.

LEMMA 2. *Suppose that $u(t)$ is a positive real valued function defined on $[t_0, \infty)$ and satisfying $(-1)^{i+1} u^{(i)}(t) > 0$ for $i = 1, 2, \cdots, n-1$, and $u^{(n)}(t) \leq 0$ on $[t_0, \infty)$. Then there is a number $\varkappa > 0$ and $t_1 \geq t_0$ such that*

$$\frac{u(t)}{u^{(n-1)}(t)} \geq \varkappa t^{n-1} \tag{6}$$

for $t \geq t_1$.

Proof. We can easily prove the lemma by argument of Heidel [4].

2. Main results

THEOREM 1. *Suppose that assumptions (i)-(iv) hold. If $\alpha > 1$ is the quotient of odd integers and*

$$\int_{t_1}^{\infty} s^{n-1} p(s)\, ds = \infty \tag{7}$$

for some $t_1 \geq 0$, then every continuable solution of (1) is oscillatory.

Proof. Let $u(t)$ be a continuable solution of (1) which is nonoscillatory on R^+. In view of Lemma 1, we may assume that there is a t_0 such that $u(t)$ and $u^{(n-1)}(t)$ are positive on $[t_0, \infty)$. From a lemma of Kiguradze [5] it follows that

$$(-1)^{i+1} u^{(n-1)}(t) \geq 0, \qquad i = 1, 2, \cdots, n-1 \tag{8}$$

on $[t_0, \infty)$. Hence there exists a finite limit $\lim_{t \to \infty} u(t) = u(\infty)$. If n is even, then $u'(t) \geq 0$ by (8) and hence $u(\infty) > 0$, or $u(\infty) = 0$. Suppose that $u(\infty) > 0$. Then there exists $t_1 \geq t_0$ such that

$$h(u(t), \cdots, u^{(n-1)}(t)) > 0, \tag{9}$$

on $[t_1, \infty)$. It follows from (1) and (9) that

$$u^{(n)}(t) + p(t)(u(t))^\alpha g(u'(t), \cdots, u^{(n-1)}(t)) < 0 \tag{10}$$

on $[t_1, \infty)$. Multiplying (10) by $t^{n-1}/(u(t))^\alpha$ and integrating it from t_1 to t, we get

$$\int_{t_1}^{t} \frac{s^{n-1} u^{(n)}(s)}{(u(s))^\alpha} \, ds + k \int_{t_1}^{t} s^{n-1} p(s) \, ds \leq 0, \tag{11}$$

which, after successive integrations by parts, yields

$$\begin{aligned} &\frac{t^{n-1} u^{(n-1)}(t)}{(u(t))^\alpha} - (n-1) \frac{t^{n-2} u^{(n-2)}(t)}{(u(t))^\alpha} + \cdots + (-1)^{n-1} \frac{(n-1)!\, u(t)}{(u(t))^\alpha} \\ &\quad + \alpha \int_{t_1}^{t} \frac{u'(s)}{(u(s))^{\alpha+1}} \{s^{n-1} u^{(n-1)}(s) - (n-1) s^{n-2} u^{(n-2)}(s) + \cdots \\ &\quad \cdots + (-1)^{n-1} (n-1)!\, u(s)\} \, ds + k \int_{t_1}^{t} s^{n-1} p(s) \, ds \leq c, \end{aligned} \tag{12}$$

where c is a constant. Using (8) and the boundedness of $u(t)$, we conclude from (12) that

$$k \int_{t_1}^{\infty} s^{n-1} p(s) \, ds < \infty,$$

which contadicts (7). Therefore, we must have $u(\infty) = 0$. This completes the proof.

THEOREM 2. *Suppose that assumptions* (i)-(iv) *hold. If* $0 < \alpha < 1$ *is the quotient of odd integers and*

$$\int_{t_1}^{\infty} s^{(n-1)\alpha} p(s) \, ds = \infty \tag{13}$$

for some $t_1 \geq 0$, *then the conclusion of Theorem 1 holds.*

Proof. Suppose that $u(t)$ is a continuable solution of (1) which is non-oscillatory on R^+. As before, we may assume that there is $[t_0, \infty)$ such that $u(t)$ and $u^{(n-1)}(t)$ are positive on $[t_0, \infty)$. As in the proof of Theorem 1, the equation (10) is then valid. Multiplying (10) by $(u^{(n-1)}(t))^{-\alpha}$ and integrating if from t_1 to t, we obtain

$$(14) \qquad \int_{t_1}^{t} \frac{u^{(n)}(s)\, ds}{(u^{(n-1)}(s))^{\alpha}} + k \int_{t_1}^{t} p(s) \left(\frac{u(s)}{u^{(n-1)}(s)} \right)^{\alpha} ds \leq 0 .$$

Since $u^{(n)}(t) < 0$ on $[t_1, \infty)$, Lemma 2 and an integration of the first term in (14) yield

$$(15) \qquad \left. \frac{(u^{(n-1)}(s))^{1-\alpha}}{1-\alpha} \right|_{t_1}^{t} + K \int_{t_1}^{t} s^{(n-1)\alpha} p(s)\, ds \leq 0 ,$$

where K is a positive constant. But (15) is impossible as $t \to \infty$. This contradiction shows that $u(t)$ is oscillatory on R^+.

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