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On the existence of cycles of given length in integer sequences like $x_{n+1} = x_n/2$ if x_n even, and $x_{n+1} = 3x_n + 1$

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Teoria dei numeri. — *On the existence of cycles of given length in integer sequences like $x_{n+1} = x_n/2$ if x_n even, and $x_{n+1} = 3x_n + 1$ otherwise* (*). Nota di CORRADO BÖHM e GIOVANNA SONTACCHI, presentata (**) dal Corrisp. B. DE FINETTI.

RIASSUNTO. — Fino ad ora non è stato possibile provare la validità della seguente nota congettura: dato un qualsiasi intero positivo x_0, la sequenza relativa ad esso definita nel titolo contiene il valore 1. Un primo passo in direzione della dimostrazione della validità della congettura potrebbe consistere nello stabilire che non esiste nessun ciclo eccetto $1, 4, 2, 1, \ldots$. Nella prima parte gli autori forniscono una procedura effettiva per decidere se esistono cicli di lunghezza data relativamente alle sequenze di razionali così definite: $x_{n+1} = ax_n + b$ se $p(x_n)$ è vera, $x_{n+1} = cx_n + d$ altrimenti, dove x_0, a, b, c, d sono razionali e p è un predicato ricorsivo. In particolare, riguardo alla congettura, propongono una nuova formulazione, più generale, e derivano, nella seconda parte, una limitazione sul valore assoluto dei termini appartenenti ad un ciclo di lunghezza data.

INTRODUCTION

A conjecture, erroneously attributed to S. Ulam [6], says that, starting from any positive integer x_0 and computing the integer sequence

$$x_{n+1} = \begin{cases} x_n/2 & \text{if } x_n \text{ even} \\ 3x_n + 1 & \text{if } x_n \text{ odd} \end{cases}$$

there exists an integer l such that $x_l = 1$.

The conjecture has been verified for some millions integers [1], [3], but, to our knowledge, nobody settled it up to the present.

The content of the conjecture may be also presented in a number of other ways [2], [4].

The conjecture circulated among computer scientists some years ago as a hard statement to prove by some induction arguments inside fixed point theory of recursive functions. It would be interesting to see how much number theory is relevant to that statement.

An algebraic version of the conjecture is: any positive integer p is obtainable starting from 1 and applying a finite numbers of times the unary operators $\lambda x.2x$ and $\lambda x.(x-1)/3$. Similar results for slightly different operators have been found [5], but the conjecture remained unproved.

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(**) Nella seduta dell'11 marzo 1978.

The Authors' research followed two directions:

i) What can be proved by splitting the conjecture in two parts and focusing only one part of it.

ii) How should the conjecture be modified if we change the basic domain from the set of positives **P** to the full set **Z** of integers.

Relatively to i) we notice that the conjecture is equivalent to the simultaneous verifying of two conditions for every initial value of x_0:

I) The only positive cyclic sequence is 1, 4, 2, 1 ,⋯ .

II) Every sequence is bounded.

Relatively to ii) we remark that taking into account zero and negative numbers some more cyclic sequences occur:

$0, 0, 0, \cdots$

$-1, -2, -4, -1, \cdots$

$-5, -14, -7, -20, -10, -5, \cdots$

$-17, -50, -25, -74, -37, -110, -55, -164, -82,$
$-41, -122, -61, -182, -91, -272, -136,$
$-68, -34, -17, \cdots$

The Authors think that the conjecture should be consequently modified into:

I′) No cycle occurs for $x_0 < -272$ and $x_0 > 4$.

II′) Every sequence is bounded in absolute value.

The fact that by the natural generalization from **P** to **Z** automatically the joined sequence (for $x_0 > 0$)

$$x_{n+1} = x_n/2 \quad \text{if } x_n \text{ even and } x_{n+1} = 3x_n - 1 \quad \text{otherwise}$$

comes into consideration, suggested us a further generalization, even by limiting us to consider property I or I′ only.

We limited us therefore to study the sequences of rational numbers (1) (the set of which is denoted by **Q**) defined by

$$(0) \quad x_{n+1} = ax_n + b \quad \text{if } p(x_n) \text{ is true and } x_{n+1} = cx_n + d \quad \text{otherwise}$$

where p is any recursive predicate, and $x_0, a, b, c, d \in$ **Q**.

(1) The extension from **Q** to **R** could also be taken into account by applying to the intuitionistic theory of real numbers without losing the constructive character (as remarked by W. Gross).

In order to acquire a better insight of the conjecture I or I', we present here a constructive result about the existence of cycles of given length in the original sequence deduced almost exclusively from the study of (0). It is to be hoped that further results could be achieved by a more profound usage of number theory.

1 - Let us consider the class **L** of the functions from **Q** to **Q** which contains all and only the functions $l(x)$ thus defined:

$$l(x) = \begin{cases} f(x) = ax + b & \text{if } p(x) \text{ is true} \\ g(x) = cx + d & \text{otherwise} \end{cases}$$

where a, b, c, d are rationals and p is any recursive predicate.

Once $n \in \mathbf{P}$ has been fixed, it is possible to determine in correspondence to each $l \in \mathbf{L}$, the set $\{l_j(x)\}_{1 \le j \le 2^n}$ of all the functions defined by the application, iterated n times, of f and g (we have, obviously, for each $l \in \mathbf{L}$, $l^n(x) \in \{l_j(x)\}_{1 \le j \le 2^n}$).

For the generic composition of f and g, the following is valid:

PROPOSITION 1. *$\forall x \in \mathbf{Q}$ and $\forall (n_0, n_1, \cdots, n_m) \in \mathbf{N}^{m+1}$ ($\mathbf{N}$ = set of naturals) we have*:

$$f^{n_m} g f^{n_{m-1}} g \cdots f^{n_1} g f^{n_0}(x) =$$

$$c^m a^{n-m} x + b \frac{a^{n_m} - 1}{a - 1} + \sum_{j=1}^{m} a^{\sum_{t=j}^{m} n_t} c^{m-j} \left(bc \frac{a^{n_{j-1}} - 1}{a - 1} + d \right)$$

where

$$n = m + \sum_{j=0}^{m} n_j .$$

(*the proof by induction on m is substantially obvious*).

From the preceding definitions and from Proposition 1 it then follows:

PROPOSITION 2. *$\forall l_j \in \{l_j\}_{1 \le j \le 2^n}$, $\exists!\, x \in \mathbf{Q}$, such that $l_j(x) = x$; therefore $\forall n \in \mathbf{P}$ and $\forall l \in \mathbf{L}$, there exists at most 2^n rationals such that $l^n(x) = x$.*

PROPOSITION 3. *$\forall x \in \mathbf{Q}$, $\forall l \in \mathbf{L}$ and $\forall n \in \mathbf{P}$, if $l^n(x) = x$ we have $p_1 < x < p_2$, where p_1 and p_2 are elements of $\mathbf{Q}$ depending from a, b, c, d, n* [2].

From the preceding propositions we obtain:

l) an effective procedure is given to construct the rationals which can belong to cycles of a defined length for a fixed $l \in \mathbf{L}$;

(2) The explicit dependence of p_1 and p_2 from a, b, c, d, n is here omitted, while in the particular case of the sequence of the title will expressed in the Proposition 6.

II) since the overmentioned rationals are in finite number (at most 2^n), it is possible to proceed to an effective test of their belonging to such cycles;

III) the absolute value of such rationals is bounded.

The Proposition 3 (and, consequently, observation III) achieves a particularly simple form when $a, c > 0$. In fact, if we consider the interval $\left(\min\left\{\frac{b}{1-a}, \frac{d}{1-c}\right\}, \max\left\{\frac{b}{1-a}, \frac{d}{1-c}\right\}\right)$, it is possible to prove that if $a, c > 1$ or $a, c < 1$ the rationals belonging to a cycle are necessarily internal to it while if $a > 1$ and $c < 1$ are external to it.

2 – Let us now particularize what has been observed with respect to the generic $l \in \mathbf{L}$ to the case of the function u from $\mathbf{Z}$ to $\mathbf{Z}$ so defined:

$$u(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ (3x+1)/2 & \text{otherwise} \end{cases}$$

(it is to be noticed that it is equivalent to consider the function u or the sequence of the conjecture). If in Proposition 1 $f(x) = x/2$ and $g(x) = (3x+1)/2$, we obtain for the function u the following formula of iteration:

PROPOSITION 4. *$\forall x \in \mathbf{Z}$ and $\forall \langle n_0, n_1, \cdots, n_m \rangle \in \mathbf{N}^{m+1}$ we have:*

$$u^n(x) = \frac{3^m x + \sum_{k=0}^{m-1} 3^{m-k-1} 2^{v_k}}{2^n}$$

where

$$v_i = i + \sum_{j=0}^{i} n_j \qquad \textit{and} \qquad n = v_m .$$

The following is also valid:

PROPOSITION 5. *$\forall x \in \mathbf{Z}$ we have: $\exists n \in \mathbf{P} \mid u^n(x) = x$ if and only if $\exists \langle v_0, v_1, \cdots, v_m \rangle \in \mathbf{N}^{m+1}$ with $v_{i-1} < v_i$ for $1 \leq i < m$ and $v_m = n$ such that*

$$x = \frac{\sum_{k=0}^{m-1} 3^{m-k-1} 2^{v_k}}{2^n - 3^m} . \qquad (*)$$

Proof. It may be observed firstly that $f(x)$ is an integer only if x is even and $g(x)$ is an integer only when x is odd: thus if an alternance of f and g gives an integer (in our case the integer x into itself) then this is precisely that alternance envisaged by the function u. Now if x belongs to a cycle it is necessarily of the form (*) (for Proposition 4) and, viceversa, if x is a integer of the form (*) it presents, with respect to the iterated application of u, exactly that alternance of f and g that is implicitly defined by the formula (*).

It may be observed that the Proposition 5 constitutes an effective refinement of Proposition 2, since it enables the search for integers belonging to a cycle of length n (or a submultiple of n) to be brought back to the search for integers in the set of the 2^n rationals of the form (*), corresponding to the 2^n functions u_j defined in analogy to the functions l_j. It is moreover possible to find a simple limitation on the absolute value of the integers belonging to cycles of fixed length. The following in fact is valid:

PROPOSITION 6. $\forall x \in \mathbf{Z} \mid u^n(x) = x$ *we have* $|x| < 3^n$.

Proof. From Proposition 5 it follows $|x| \leq \sum_{k=0}^{m-1} 3^{m-k-1} 2^{v_k}$ where the v_i satisfy the usual conditions. Then we have:

$$|x| \leq \sum_{k=0}^{m-1} 3^{m-k-1} 2^{n-m-k} = 2^{n-m}(3^m - 2^m) < 3^n .$$

Finally let $\mathbf{U} = \{x \in \mathbf{Z} \mid \exists n \in \mathbf{P} : u^n(x) = 1\}$; it is possible by analogy with Proposition 5 to prove:

PROPOSITION 7. $\forall x \in \mathbf{Z}$ *we have*: $x \in \mathbf{U}$ *if and only if*

$\exists \langle v_0, v_1, \cdots, v_m \rangle \in \mathbf{N}^{m-1}$ *with* $v_{i-1} < v_i$ *for* $1 \leq i < m$ *such that*

$$(**) \qquad x = \frac{2^{v_m} - \sum_{k=0}^{m-1} 3^{m-k-1} 2^{v_k}}{3^m} .$$

The verification of former Ulam's conjecture can thus be brought back to the proof that each natural can be expressed in the form (**).

In particular the naturals $1, 4 \times 1 + 1 = 5, 4 \times 5 + 1 = 21, 4 \times 21 + + 1 = 85$ etc., verify, for example, the condition envisaged by the conjecture, since it is possible to write them in the form $(2^{2p} - 1)/3$.

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