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4. ENERGY-MOMENTUM TENSORS FOR THE SHELLS

According to our hypotheses, the energy-momentum tensor for the inner shell has the form

$$(20) \qquad \overline{C}_{ik} = (\overline{\mu}_0 c^2 + \tfrac{1}{2} r_0 \rho_0^2)\, \bar{u}_i \bar{u}_k + \sum_{\alpha}{}_1^3 \bar{q}(\alpha)\, \overline{V}_i^{(\alpha)} \overline{V}_k^{(\alpha)} \qquad \text{(inner shell)}$$

$\bar{u}_i$ is the unit velocity four-vector; $\overline{V}_i^{(\alpha)}$ form a triad of orthonormal spatial vectors spanning the hyperspace orthogonal to $\bar{u}_i$; $\bar{q}(\alpha)$ are the principal stresses, and $\overline{\mu}_0 + \frac{1}{2} r_0 \rho_0^2/c^2$ is the proper surface mass density. It includes matter density $\overline{\mu}_0$ as well as electrostatic energy-mass density. In a similar way the expressions for the principal stresses $\bar{q}(\alpha)$ include a mechanical stress term $\bar{p}(\alpha)$, to be determined, as well as a known maxwellian stress term:

$$(21) \qquad \bar{q}(1) \equiv \bar{p}(1) - \tfrac{1}{2} r_0 \rho_0^2 \quad , \quad \bar{q}(2) \equiv \bar{p}(2) + \tfrac{1}{2} r_0 \rho_0^2 ,$$
$$\bar{q}(3) \equiv \bar{p}(3) + \tfrac{1}{2} r_0 \rho_0^2 .$$

The inner shell is at rest with respect to S_3. That is, the time tracks of its material elements are the trajectories of S_3 labelled by $r = r_0$. Following such a remark, one can write:

$$(22) \qquad \bar{u}^i = \underset{(4)}{\vartheta^i}(r_0) / [-\underset{(4)}{\vartheta^h}\, \underset{(4)}{\vartheta_h}(r_0)]^{\frac{1}{2}} =$$
$$= \left(0 \, , \, \frac{\bar{w}(3\,,2)/r_0 c}{\sqrt{1 - \bar{w}(3\,,2)^2/c^2}} \, , \, 0 \, , \, \frac{1}{\sqrt{1 - \bar{w}(3\,,2)^2/c^2}}\right).$$

The remaining vectors $\overline{V}_i^{(\alpha)}$ are then uniquely determined by orthonormality with respect to $\bar{u}^i$.

The outer shell is rotating with respect to S_3, namely with respect to the inner shell, with a known standard angular velocity ω, of magnitude $W = \omega R_0 A$. Hence:

$$(22') \qquad \hat{u}^i = \left(0 \, , \, \frac{\omega/c}{\sqrt{1 - W^2/c^2}} \, , \, 0 \, , \, \frac{A}{\sqrt{1 - W^2/c^2}}\right),$$

(*) Nella seduta del 14 gennaio 1978.

$u^{\hat{i}}$ being the unit velocity four-vector. According to the remarks of section (3), a current flows in the outer shell, and we have to assume for it a perfectly conducting material structure. The question arises what expression must be assumed for its energy-momentum tensor $C_{\hat{i}\hat{k}}$. For this purpose, let us put ourselves in the frame S comoving with the outer shell. With respect to S, on the shell, there is a surface charge density

$$\tilde{\rho} \equiv -u^{\hat{i}} J_{\hat{i}} = \frac{\omega/c}{(1 - W^2/c^2)^{\frac{1}{2}}} \cdot$$

$$\cdot \left[\frac{\bar{w}(2,3)}{c} \left(\frac{R_0}{r_0} \right)^{-2b^2} \frac{1 - v(1,2)\,\bar{w}(2,3)/c^2}{[(1 - \bar{w}^2(2,3)/c^2)\cdot(1 - w^2(2,3)/c^2)]^{\frac{1}{2}}} \right] R_0\, \rho_0$$

as well as a surface current density $\tilde{j}_{\hat{i}} = (g_{ik} + u_i u_k) J^k$. *We consider the outer shell realized, via a limiting process, from a shell of finite thickness* ε, *with a volume charge density* $\tilde{\rho}\varepsilon$ *and in which the current* $\tilde{j}_{\hat{i}}\varepsilon$ *flows. It is easy to see that, for* $\varepsilon \to 0+$, *only* $\tilde{\rho}$ *gives rise to an e.m. contribution to the energy-momentum tensor* $C_{\hat{i}\hat{k}}$, *and we can write*

$$C_{\hat{i}\hat{k}} = (\mu_0 c^2 + \tfrac{1}{2} R_0 \tilde{\rho}^2)\, u_{\hat{i}} u_{\hat{k}} + \sum_{\alpha}{}_1^3\, q(\alpha)\, V_{\hat{i}}^{(\alpha)} V_{\hat{k}}^{(\alpha)} \qquad \text{(outer shell)}: \tag{23}$$

$u_{\hat{i}}$ is the unit velocity four-vector given by (22′); $V_{\hat{i}}^{(\alpha)}$ form a triad of orthonormal spatial vectors orthogonal to $u_{\hat{i}}$; μ_0 is the proper mass density of the outer shell; $q(\alpha)$ are the principal stresses given by:

$$q(1) \equiv p(1) - \tfrac{1}{2} R_0 \tilde{\rho}^2 \quad , \quad q(2) \equiv p(2) + \tfrac{1}{2} R_0 \tilde{\rho}^2 , \tag{24}$$
$$q(3) \equiv p(3) + \tfrac{1}{2} R_0 \tilde{\rho}^2 ,$$

$p(\alpha)$ being mechanical stress terms.

5. HIGHER ORDER GRAVITATIONAL JUNCTION CONDITIONS

(20) and (23) give the tensor components $\overline{C}_{ik}$ and $C_{\hat{i}\hat{k}}$ following from the structural hypotheses we have assumed about the shells. On the other hand such components are obtained too, as functions of the undetermined parameters b, B, $v(1,2)$, $w(2,3)$, imposing the following higher order gravitational junction conditions on $\overline{\Sigma}$ and Σ, [2]:

$$\tfrac{1}{2}\{g^{lm}[g_{ik,l}]\,\bar{n}_m - \bar{n}^m([g_{im,k}] + [g_{mk,i}]) + g^{lm}[g_{lm,i}]\,\bar{n}_k\}_{r_0} = \tag{25}$$
$$= -\chi(\overline{C}_{ik} - \tfrac{1}{2} g_{ik}(r_0)\,\overline{C}_i^i) ,$$

$$\tfrac{1}{2}\{g^{\hat{l}\hat{m}}[g_{\hat{i}\hat{k},\hat{l}}]\,n_{\hat{m}} - n^{\hat{m}}([g_{\hat{i}\hat{m},\hat{k}}] + [g_{\hat{m}\hat{k},\hat{i}}]) + g^{\hat{l}\hat{m}}[g_{\hat{l}\hat{m},\hat{i}}]\,n_{\hat{k}}\}_{R_0} = \tag{25'}$$
$$= -\chi(C_{\hat{i}\hat{k}} - \tfrac{1}{2} g_{\hat{i}\hat{k}}(R_0)\,C_{\hat{i}}^{\hat{i}}) .$$

Eliminating $\overline{C}_{ik}$ *and* $C_{\hat{i}\hat{k}}$ *between* (20), (23) *and* (25), (25′) *respectively, we obtain ten independent functional relations, which give the ten unknown quantities*

$\bar{q}(\alpha)$, $q(\alpha)$, $v(1, 2)$, $w(2, 3)$, b, B, *as functions of the data of the problem* $\bar{\mu}_0$, μ_0, ρ_0, r_0, R_0, W.

Four relations at once yield:

$$(26) \qquad \bar{q}(1) = 0 \quad , \quad \bar{q}(3) = 0 \quad , \quad q(1) = 0 \quad , \quad q(3) = 0 .$$

The null value for $\bar{q}(1)$ and $q(1)$ follows from the relations $\bar{C}_{ik}\bar{n}^i = 0$ and $C_{\hat{i}\hat{k}}n^{\hat{i}} = 0$, algebraic consequences of (25) and (25′) respectively. They state nothing but that $\bar{\Sigma}$ and Σ are the histories of the shells. The null value for $\bar{q}(3)$ and $q(3)$ follows from the relations $\bar{q}(1) + \bar{q}(3) = 0$ and $q(1) + q(3) = 0$, which hold for the sources of the Levi-Civita's metric (4) with $R(r) = r$.

From (26) we obtain the radial and axial stress terms acting in the shells: $\bar{p}(1) = -\bar{p}(3) = \frac{1}{2} r_0 \rho_0^2$, *for the inner shell*; $p(1) = -p(3) = \frac{1}{2} R_0 \bar{\rho}^2$, *for the outer shell*. For both, a pressure is present in the radial direction, and a tension in the axial direction. In each shell, such stress terms hold together the material elements against the repulsive coulomb interaction acting between them.

The form rather involved of the remaining six relations implies that it will be almost impossible to find solutions for b, B, $\bar{q}(2)$, $q(2)$, $v(1, 2)$, $w(2, 3)$ explicitely expressed as functions of the data of the problem. *However it is possible to display such solutions as simple functions of physically measurable quantities*. For these latter we mean the *magnitudes that the standard gravitational fields, in* A_1, A_2, A_3, *respectively*,

$$G'_{i'} = -\tfrac{1}{2} c^2 \partial_{i'} \ln(-g_{4'4'}) \quad , \quad G_i = -\tfrac{1}{2} c_i \partial_i \ln(-g_{44}) ,$$
$$\hat{G}_{\hat{i}} = -\tfrac{1}{2} c^2 \partial_{\hat{i}} \ln(-g_{\hat{4}\hat{4}})$$

assume on the hypersurfaces $\bar{\Sigma}$ *and* Σ [(1)] [3], [4],

Following such remarks, we obtain for the *Levi-Civita's masses* b and B:

$$(27) \qquad b = -\left[\tfrac{1}{2}\chi r_0 (\bar{p}(2) - \bar{\mu}_0 c^2) + r_0 (G'(r_0) + G(r_0))/c^2\right]^{\frac{1}{2}},$$

$$(27') \qquad B = -\left[b^2 + \tfrac{1}{2} A^2 (R_0/r_0)^{2b^2} \chi R_0 (p(2) - \mu_0 c^2) + \right.$$
$$\left. + A R_0 (R_0/r_0)^{b^2} (\hat{G}(R_0) - G(R_0))/c^2\right]^{\frac{1}{2}}.$$

Their negative value corresponds to the attractive character of the gravitational fields G and $\hat{G}$ in the regions A_2 and A_3 respectively. The physical

(1) That is, on $\bar{\Sigma}: G'(r_0) = \lim_{\varepsilon \to 0+} [g^{1'1'} G'^2_{1'}(r_0 - \varepsilon)]^{\frac{1}{2}}$,

and

$$G(r_0) = \lim_{\varepsilon \to 0+} [g^{11} G^2_1(r_0 + \varepsilon)]^{\frac{1}{2}}; \text{ on } \Sigma : G(R_0) = \lim_{\varepsilon \to 0+} [g^{11} G^2_1(R_0 - \varepsilon)]^{\frac{1}{2}},$$

and

$$\hat{G}(R_0) = \lim_{\varepsilon \to 0+} [g^{\hat{1}\hat{1}} \hat{G}^2_{\hat{1}}(R_0 + \varepsilon)]^{\frac{1}{2}}.$$

interpretation of the constants b and B comes out particularly clearly in the *newtonian approximation.* To first order in the gravitational constant χ, one obtains:

$$b \cong -\tfrac{1}{2} r_0 (\bar{\mu}_0 c^2 + \tfrac{1}{2} r_0 \rho_0^2) \chi , \tag{28}$$

$$B \cong -\tfrac{1}{2} r_0 (\bar{\mu}_0 c^2 + \tfrac{1}{2} r_0 \rho_0^2) \chi - \tfrac{1}{2} R_0 \mu_0 c^2 (1 + R_0 \omega^2/c^2) \chi . \tag{28'}$$

Hence $-2\, b/\chi$ *measures the proper energy density per unit length of the inner shell. The constant* $-2\, B/\chi$, *besides the previous term, gives the proper material energy density as well as the rotational energy density per unit length of the outer shell.*

Such an interpretation seems to be in accordance with the relative definition and decomposition of the total gravitational field energy proposed by C. Cattaneo in [5].

The relation for the magnitude, on Σ, of the linear standard velocity of S_2 with respect to S_3, $w(2, 3) = R_0 A^2 \omega(2, 3)$, yields:

$$\frac{w(2, 3)/c}{1 - w^2(2, 3)/c^2} = \tag{29}$$

$$= \chi R_0 \left(\frac{R}{r_0}\right)^{2b^2} \frac{\mu_0 c^2 + \tfrac{1}{2} R_0 \tilde{\rho}^2 + q(2)}{[1 - 2 R_0 A (R_0/r_0)^{b^2} \cdot G(R_0)/c^2]} \cdot \frac{W/c}{1 - W^2/c^2} .$$

Hence $w(2, 3)$ *vanishes only when* W *or* χ *does. That is, the motion of the outer shell and the existence of a gravitational coupling induce a mutual rotation between the frames* S_2 *and* S_3. This effect, which gives rise to Coriolis-type forces, is the well known *Thirring effect,* [7].

In the *newtonian approximation* one obtains:

$$\omega(2, 3) \cong \mu_0 c^2 R_0 \omega\chi . \tag{30}$$

that is, just a Thirring-like value.

In a similar way one finds the magnitude, on $\bar{\Sigma}$, of the linear standard velocity of S_1 with respect to S_2, $v(1, 2) = r_0 \omega(1, 2)$:

$$\frac{v(1, 2)/c}{1 - v^2(1, 2)/c^2} = \chi r_0 \frac{\bar{\mu}_0 c^2 + \tfrac{1}{2} r_0 \rho_0^2 + \bar{q}(2)}{1 - 2 r_0 G'(r_0)/c^2} \cdot \frac{\bar{w}(2, 3)/c}{1 - \bar{w}^2(2, 3)/c^2} . \tag{31}$$

Where $\bar{w}(2, 3) = r_0 \omega(2, 3) = r_0 w(2, 3)/R_0 A^2$, $w(2, 3)$ being given by (29), is the magnitude, on $\bar{\Sigma}$, of the linear standard velocity of S_2 with respect to the inner shell.

Again, $v(1, 2)$ *vanishes only when* W *or* χ *does. The rotation of the outer shell not only induces the mutual rotation between* S_2 *and* S_3, *but also between* S_1 *and* S_2. *This is a new, vanishingly small, effect, absent in the newtonian approximation.*

To second order in the gravitational constant χ, *one finds*:

$$(31') \qquad \omega(1, 2) \cong -r_0 R_0 \mu_0 c^2 (\bar{\mu}_0 c^2 + \tfrac{1}{2} r_0 \rho_0^2) \omega\chi^2 .$$

Hence in a newtonian approximation the stationary frames S_1 *and* S_2 *are at relative rest. They define a single stationary frame* S. *which rotates with respect to the inner shell with linear velocity* $\bar{w}(2, 3) \cong \mu_0 c^2 r_0 R_0 \omega\chi$. *Notice that such a rotation rate does not depend on the mass distribution of the inner shell.*

From $w(3, 2) = -w(2, 3)$ and $v(2, 1) = -v(1, 2)$, one obtains, by means of (15) the magnitude, on Σ, of the linear standard velocity of the inner shell with respect to S_1 :

$$(32) \qquad v(3, 1) = \frac{\bar{w}(3, 2) + v(2, 1)}{1 + \bar{w}(3, 2) v(2, 1)/c^2} .$$

To first order in the gravitational constant χ, (32) reduces to: $v(3, 1) \cong \cong \bar{w}(3, 2) \cong -\mu_0 c^2 r_0 R_0 \omega\chi$, in accordance with our previous considerations.

Finally for the tangential stresses $\bar{q}(2)$ and $q(2)$ acting in the inner and in the outer shell respectively, one obtains:

$$(33) \qquad \tfrac{1}{2} \chi r_0 \bar{q}(2) = -\tfrac{1}{2} \chi r_0 (\bar{\mu}_0 c^2 + \tfrac{1}{2} r_0 \rho_0^2) + $$
$$+ \frac{(1 - \bar{w}^2(2, 3)/c^2) v^2(1, 2)/c^2}{(1 + \bar{w}^2(2, 3)/c^2)(1 - v^2(1, 2)/c^2)} (1 - 2 r_0 G'(r_0)/c^2)$$
$$+ \frac{1 - \bar{w}^2(2, 3)/c^2}{1 + \bar{w}^2(2, 3)/c^2} r_0 (G(r_0) - G'(r_0))/c^2 ,$$

$$(33') \qquad \tfrac{1}{2} \chi R_0 q(2) = -\tfrac{1}{2} \chi R_0 (\mu_0 c^2 + \tfrac{1}{2} R_0 \bar{\rho}^2) +$$
$$+ \frac{(1 - W^2/c^2) w^2(2, 3)/c^2}{(1 + W^2/c^2)(1 - w^2(2, 3)/c^2)} \left(\frac{R_0}{r_0}\right)^{-2b^2} \left[1 - R_0 A \left(\frac{R_0}{r_0}\right) \frac{G(R_0)}{c^2}\right]$$
$$+ \frac{1 - W^2/c^2}{1 + W^2/c^2} A^{-1} R_0 (\hat{G}(R_0) - G(R_0))/c^2 .$$

To first order in the gravitational constant χ, *such expressions yield*: $\bar{p}(2) \cong -\frac{1}{2} r_0 \rho_0^2$, $p(2) \cong -\mu_0 R_0^2 \omega^2$, *respectively*. In the former one recognizes the usual mechanical tension balancing the maxwellian tangential pressure acting in the inner shell. The latter represents the usual centrifugal tensions.

6. Electric and magnetic fields

According to our previous considerations, the rotation of the outer shell, with respect to the inner one, induces a mutual rotation among all the frames S_α. The constants $w(2, 3)$ and $v(1, 2)$ which describe such *Thirring-like effect*, are given by (29) and (31). Their general behaviour was discussed qualitatively in the previous section. From $v(1, 2)$ and $w(2, 3)$, by means of (18), (18′), we can obtain the values of the pseudo-scalars Ψ, Θ, Δ, which give the e.m. tensor components $F^{1'2'}$, F^{14}, $F^{\hat{1}\hat{4}}$, respectively. Then the problem of finding the e.m. fields in each region A_α can be considered solved. We give the results in each region A_α:

A_1, *the inner region.*

With respect to the static frame S_1, a *magnetic compass* would experience an axial magnetic field $H_{3'}$:

$$H_{3'} = \Psi(-g_{4'4'})^{-\frac{1}{2}} = \rho' \left(\frac{v(3, 1)}{c} - \frac{v(2, 1)}{c} \right) \frac{1 + hr_0^2}{1 + hr'^2} \tag{34}$$

everywhere regular but not constant. It attains its maximum value on the symmetry axis, $r' = 0$, its minimum on $\overline{\Sigma}$, $r' = r_0$. Hence there is a radial magnetic pressure which decreases going from the axis to the hypersurface $\overline{\Sigma}$ where the inner shell evolves. The magnetic field is "held together" against the gradient of such a pressure by its own gravitational attraction. Indeed the presence of $H_{3'}$ in A_1, gives rise to a radial standard gravitational field given by $G_{i'} = -2c^2 hr' \delta_{i'}^{1'}/(1 + hr'^2)$. *Since* $G_{i'} = O(\chi^3)$, *the presence of such a gravitational field is a vanishingly small, post-newtonian, effect.* Apart the slow-varying radial dependence, the magnetic field $H_{3'}$ is the sum of two terms: $\rho' v(3, 1)/c$ and $-\rho' v(2, 1)/c$. *At once, in the former one recognizes the surface convection terms, which arises from the rotation, with linear velocity* $v(3, 1)$, *of the inner shell with respect to* S_1. While, in order to explain the presence of the other term, we need some further considerations. We shall see, in the following sub-section, that, in the intermediate region A_2, an S_2-observer experiences a radial electric field E_1. Such a field gives rise, with respect to S_1, to an axial magnetic field, that, on $\overline{\Sigma}$, values: $\mathcal{H}_{3'}(r_0) = -v(2, 1) E_1(r_0)/c(1 - v^2(2, 1)/c^2)^{\frac{1}{2}} = -\rho' v(2, 1)/c$. Just the term, whose interpretation we are looking for.

Following such a remark, it is fairly significant that (34) can be written, on $\overline{\Sigma}$, in the form

$$H_{3'} - \mathcal{H}_{3'} = \rho' \frac{v(3, 1)}{c}, \tag{34'}$$

in which one can recognize a classical junction condition for the magnetic field.

The formula (34) takes clearly into account *the interactions between inertia and electromagnetism. The constants* $v(3, 1)$ *and* $v(2, 1)$, *as we have seen in the previous section, vanish only if one ignores the gravitational coupling or when the shells are at relative rest. Only if one of such conditions holds the magnetic field* $H_{3'}$ *as well as the gravitational field it generates, vanishes.*

To first order in the gravitational constant χ, (34) yields to a particularly simple result:

$$H_{3'} \cong -\mu_0 c^2 R_0 r_0 \frac{\rho_C \omega}{c} \chi. \tag{35}$$

Hence in the newtonian approximation only the convenction current term contributes to the magnetic field. The field is constant, the inner region A_1 is flat, the frame S_1 *is inertial.*

The interpretation of the field (35) is remarkably simple. In the newtonian approximation, according to our results, the inner shell rotates, with respect to the inertial frame S_1, with the Thirring angular velocity $\omega(3, 1) \cong -\mu_0 r^2 R_0 \omega\chi$. Its relative surface charge ρ' reduce to ρ_0, and we thus expect to find within it a constant axial magnetic field $H^*_{3'} = -\mu_0 c^2 R_0 r_0 \rho_0 \omega\chi/c$. *Just the field* (35) *we have found.*

The magnetic field (34) is the field an S_1-observer would experience in the inner region A_1. The question arises of what e.m. fields an observer comoving with the inner shell would experience in A_1. Such fields are easily obtained by means of the transformation law for the tensor components $F^{i'k'}$:

$$F^{\hat{1}\hat{4}}(A_1) = \Theta^{\hat{4}}_{2'} F^{1'2'} \quad , \quad F^{\hat{1}\hat{2}}(A_1) = \Theta^{\hat{2}}_{2'} F^{1'2'}. \tag{36}$$

$F^{\hat{1}\hat{4}}(A_1)$ and $F^{\hat{1}\hat{2}}(A_1)$ give rise to an electric and to a magnetic field respectively. Their expressions, which can be easily worked out, are omitted here, I shall limit myself to observing that to first order in the gravitational constant χ, (36) yields:

$$E_{\hat{1}}(A_1) \cong 0 \quad , \quad H_{\hat{3}}(A_1) \cong -\mu_0 c^2 R_0 r_0 \frac{\rho_0 \omega}{c} \chi . \tag{36'}$$

Hence an observer comoving with the inner shell experiences in A_1 the same magnetic field there experienced by an S_1-observer.

A_2, *the intermediate region.*

With respect to the static frame S_2, one experiences a radial electric field E_1:

$$E_1 = \rho r_0 [1 - v(1, 2)\, \varpi(2, 3)/c^2] \left(\frac{r}{r^0}\right)^{-b} \frac{1 - H}{[1 - H(r/r_0)^{-2b}]} \frac{1}{r} . \tag{37}$$

Apart from the geometrical factor giving the radial dependence, the electric field (37) is the difference of two terms: ρ and $\rho v(1, 2)\, \varpi(2, 3)/c^2$. The *former is the surface charge density of the inner shell with respect to* S_2. The latter, *an apparent source term*, arises from the transformation of the axial magnetic field $H_{3'}$, present in A_1, in a radial electric field $\mathscr{E}_1$, on account of the relative rotation of S_1 with respect to S_2. Indeed, on Σ, one obtains:

$$\mathscr{E}_1(r_0) = v(1, 2)\, H_{3'}(r_0)/c\, (1 - v^2(1, 2)/c^2)^{\frac{1}{2}} = -\rho v(1, 2)\, \varpi(2, 3)/c^2.$$

Allow me to add here that the presence of such an apparent source term, as of other similar terms, could be clearly understood also by means of the relative formulation of electromagnetism established by P. Benvenuti [1].

On Σ, (37) can be written as

$$E_1(r_0) - \mathscr{E}_1(r_0) = \rho .$$

That is, in the form of a classical junction condition for the normal discontinuity of the electric field through a charged thin shell.

If the shells were at relative rest, or if we did not take into account the gravitational interaction, E_1 *should become, apart from a geometrical factor, the usual electrostatic field outside a charged cylindrical shell. Otherwise its strength increases by the positive apparent source term* $-\rho v(1, 2)\,\bar{w}(2, 3)/c^2$. *Such a term represents an extremely little post-newtonian correction, small of third order in the gravitational constant* χ.

In a newtonian approximation (37) reduces to:

$$E_1 \cong \rho_0 r_0 / r \,. \tag{38}$$

A_3, *the outer region.*

With respect to the static frame S_3, one experiences a radial electric field $E_{\hat{1}}$:

$$E_{\hat{1}} = \rho_0 r_0 \frac{1 - v(1, 2)\,\bar{w}(2, 3)/c^2}{(1 - \bar{w}^2(2, 3)/c^2)^{\frac{1}{2}}(1 - w^2(2, 3)/c^2)^{\frac{1}{2}}} \left(\frac{\hat{r}}{R_0}\right)^{-B} \cdot \frac{1 - K}{A[1 - K(\hat{r}/R_0)^{-2B}]} \frac{1}{\hat{r}} \,. \tag{39}$$

Apart from the different radial dependence, (39) *is the same electric field* (37) *present in* A_2, *now expressed in the* S_3*-adapted coordinates* $(x^{\hat{i}})$. Indeed with respect to S_3, on Σ, the field (37) becomes

$$\hat{\mathscr{E}}_{\hat{1}}(r_0) = E_1(r_0)/(1 - w(2, 3)^2/c^2)^{\frac{1}{2}} = E_{\hat{1}}(r_0) \,.$$

Following such remarks *we find again that the outer shell is uncharged with respect to* S_3, (*that is, with respect to the inner shell*). Indeed (39) can be written, on Σ, in the form

$$E_{\hat{1}}(R_0) - \hat{\mathscr{E}}_{\hat{1}}(R_0) = 0 \,,$$

in which one can recognize the classical statement for the electrical neutrality of a thin shell. In agreement with such remarks, in a newtonian approximation (39) reduces to: $E_{\hat{1}} \cong \rho_0 r_0 / \hat{r}$.

7. SOME CONCLUDING REMARKS

According to Rindler [6], our problem can be considered the *Mach-equivalent view* of a charged cylindrical thin shell in uniform rotation with respect to the "*universe*".

Within such a shell, a magnetic compass, (at rest with respect to the "universe"), would experience an axial magnetic field $\bar{H}_3 = -\bar{\rho}\,\frac{\bar{r}_0 \bar{\omega}}{c}$, $\bar{\omega}$ being the rotation rate of the shell, $\bar{r}_0$ its radius, $\bar{\rho}$ its surface density of charge.

Requiring a true physical reciprocity, this same field must be experienced also if, changing our point of view, we regard the shell at rest and the universe rotating round it. Hence, by analogy, we would find a similar field within a charged shell surrounded by another coaxial massive shell rotating with respect to it. Our results seems to be in accordance with such a conjecture. This is particularly clear in a newtonian approximation, in which we obtain an axial magnetic field (*cfr.* (35), (36′)), *that agrees fully with* $\overline{H}_3$.

The " neutral " case, in which also the inner shell is supposed to be uncharged, should deserve particular attention. Of course, in its solution neither electric nor magnetic fields appear. However, from the gravitational standpoint, our neutral problem corresponds to a more general situation than those described by Thirring, Frehland, Pietronero, and it gives rise to physically meaningful results.

We deal with it in another paper.

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