Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Takashi Noiri

On S-closed subspaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **64** (1978), n.2, p. 157–162.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1978_8_64_2_157_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1978.

Topologia. — *On S-closed subspaces.* Nota di TAKASHI NOIRI, presentata (*) dal Socio E. MARTINELLI.

RIASSUNTO. — Thompson [6] ha introdotto il concetto di spazio S–chiuso. Scopo del presente lavoro è studiare alcune proprietà di un sottospazio S–chiuso, introdurre e caratterizzare gli spazi detti localmente S–chiusi.

1. INTRODUCTION

Let X be a topological space and S a subset of X. The closure of S in X and the interior of S in X will be denoted by $\mathrm{Cl}_X(S)$ and $\mathrm{Int}_X(S)$, respectively. In [1], N. Levine has defined a subset S to be *semi-open* if there exists an open set V of X such that $V \subset S \subset \mathrm{Cl}_X(V)$. Recently, T. Thompson [6] has defined a space X to be *S-closed* if for every semi-open cover $\{V_\alpha \mid \alpha \in \nabla\}$ of X there exists a finite subfamily ∇_0 of ∇ such that $X = \cup \{\mathrm{Cl}_X(V_\alpha) \mid \alpha \in \nabla_0\}$. In [4], the present Author has obtained several properties of such spaces. In this paper, in order to investigate S-closed subspaces we shall define and study a subset S-closed relative to X. Moreover, we shall define the concept of locally S-closed spaces and obtain some basic properties of such spaces. Throughout this paper spaces will always mean topological spaces on which no separation axioms are assumed unless explicitly stated. By SO (X) we shall denote the family of all semi-open sets in a space X. A subset S of a space X is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}_X(\mathrm{Cl}_X(S)) = S$ (resp. $\mathrm{Cl}_X(\mathrm{Int}_X(S)) = S$).

2. PRELIMINARIES

DEFINITION 2.1. A subset S of a space X is said to be *S-closed relative to* X if for every cover $\{V_\alpha \mid \alpha \in \nabla\}$ of S by semi-open sets in X there exists a finite subfamily ∇_0 of ∇ such that $S \subset \cup \{\mathrm{Cl}_X(V_\alpha) \mid \alpha \in \nabla_0\}$.

Remark 2.1. Every set S-closed relative to a space X is not necessarily an S-closed subspace of X, even if it is closed in X, as the following example shows.

Example 2.1. Let R be the set of real numbers, Γ the topology of countable complements of R and N the set of positive integers. Then N is S-closed relative to (R, Γ) and closed in (R, Γ), but not an S-closed subspace.

(*) Nella seduta dell'11 febbraio 1978.

DEFINITION 2.2. A subset S of a space X is called an *α-set* [2] if $S \subset \mathrm{Int}_X(\mathrm{Cl}_X(\mathrm{Int}_X(S)))$.

Remark 2.2. Every open set is an α-set and every α-set is semi-open. However, the converse implications are not necessarily true, as the following examples show. In [2], O. Njåstad called semi-open sets β-sets.

Example 2.2. Let R be the set of real numbers with the usual topology. Then $(0, 1]$ is a semi-open set of R, but not an α-set.

Example 2.3. Let $X = \{a, b, c\}$ and $\Gamma = \{\varnothing, \{a\}, \{a, c\}, X\}$. Then $\{a, b\}$ is an α-set, but not open in (X, Γ).

V. Pipitone and G. Russo [5] showed that the intersection of $A \in SO(X)$ and $V \in SO(X)$ is not necessarily semi-open in the subspace A, however it is true if A is open in X [5, Teorema 2.2]. The following lemma is a slight improvement of this result.

LEMMA 2.1. *If* A *is an α-set of a space* X *and* $V \in SO(X)$, *then* $A \cap V \in SO(A)$.

Proof. Since A is an α-set of X and $V \in SO(X)$, $A \cap V \in SO(X)$ [2, Proposition 1] and hence $A \cap V \in SO(A)$ [1, Theorem 6].

3. S-CLOSED SUBSPACES

THEOREM 3.1. *An α-set* A *of a space* X *is an* S-*closed subspace if and only if it is* S-*closed relative to* X.

Proof. Necessity. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of A and $V_\alpha \in SO(X)$ for each $\alpha \in \nabla$. Since A is an α-set, by Lemma 2.1 $A \cap V_\alpha \in SO(A)$ for each $\alpha \in \nabla$. Since A is S-closed, there exists a finite subfamily ∇_0 of ∇ such that $A = \cup \{\mathrm{Cl}_A(A \cap V_\alpha) \mid \alpha \in \nabla_0\}$. Therefore, we have $A \subset \cup \{\mathrm{Cl}_X(V_\alpha) \mid \alpha \in \nabla_0\}$. This shows that A is S-closed relative to X.

Strong sufficiency. Suppose that $A \in SO(X)$ and A is S-closed relative to X. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of A and $V_\alpha \in SO(A)$ for each $\alpha \in \nabla$. Then, we have $V_\alpha \in SO(X)$ for each $\alpha \in \nabla$ [3, Theorem 1]. Therefore, there exists a finite subfamily ∇_0 of ∇ such that $A \subset \cup \{\mathrm{Cl}_X(V_\alpha) \mid \alpha \in \nabla_0\}$. Thus, we obtain $A = \cup \{\mathrm{Cl}_A(V_\alpha) \mid \alpha \in \nabla_0\}$. This shows that A is S-closed.

THEOREM 3.2. *Let* A *and* X_0 *be subsets of a space* X *such that* $A \subset X_0 \subset X$ *and* X_0 *is an α-set. Then* A *is* S-*closed relative to the subspace* X_0 *if and only if* A *is* S-*closed relative to* X.

Proof. Necessity. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of A and $V_\alpha \in SO(X)$ for each $\alpha \in \nabla$. Then, by Lemma 2.1 $X_0 \cap V_\alpha \in SO(X_0)$ for each $\alpha \in \nabla$ and $A \subset \cup \{X_0 \cap V_\alpha \mid \alpha \in \nabla\}$. Since A is S-closed relative to X_0, there exists a finite subfamily ∇_0 of ∇ such that $A \subset \cup \{\mathrm{Cl}_{X_0}(X_0 \cap V_\alpha) \mid \alpha \in \nabla_0\}$. Therefore we have $A \subset \cup \{\mathrm{Cl}_X(V_\alpha) \mid \alpha \in \nabla_0\}$. This shows that A is S-closed relative to X.

Strong sufficiency. Let X_0 be a semi-open set of X. Suppose that $\{V_\alpha \mid \alpha \in \nabla\}$ is a cover of A and $V_\alpha \in SO(X_0)$ for each $\alpha \in \nabla$. Then, we have $V_\alpha \in SO(X)$ for each $\alpha \in \nabla$ [3, Theorem 1]. Since A is S-closed relative to X, there exists a finite subfamily ∇_0 of ∇ such that $A \subset \cup\{Cl_X(V_\alpha) \mid \alpha \in \nabla_0\}$. Therefore, we obtain $A \subset \cup\{Cl_{X_0}(V_\alpha) \mid \alpha \in \nabla_0\}$. This shows that A is S-closed relative to X_0.

COROLLARY 3.1. *Let* A *and* X_0 *be open sets of a space* X *such that* $A \subset X_0$. *Then* A *is an* S-*closed subspace of* X_0 *if and only if* A *is an* S-*closed subspace of* X.

Proof. This follows from Theorem 3.1 and Theorem 3.2.

THEOREM 3.3. *Let* A *and* B *be subsets of a space* X. *If* A *is* S-*closed relative to* X *and* B *is regular open in* X, *then* $A \cap B$ *is* S-*closed relative to* X.

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of $A \cap B$ and $V_\alpha \in SO(X)$ for each $\alpha \in \nabla$. Since $X - B$ is regular closed, we have $X - B \in SO(X)$ and

$$A \subset [\cup\{V_\alpha \mid \alpha \in \nabla\}] \cup (X - B)\,.$$

Since A is S-closed relative to X, there exists a finite subfamily ∇_0 of ∇ such that $A \subset [\cup\{Cl_X(V_\alpha) \mid \alpha \in \nabla_0\}] \cup (X - B)$. Therefore, we obtain $A \cap B \cup \cup\{Cl_X(V_\alpha) \mid \alpha \in \nabla_0\}$. This shows that $A \cap B$ is S-closed relative to X.

COROLLARY 3.2. *If* X *is an* S-*closed space and* A *is a regular open set of* X, *then* A *is an* S-*closed subspace of* X.

Proof. This follows from Theorem 3.1 and Theorem 3.3.

COROLLARY 3.3. *Let* A *be a set* S-*closed relative to a space* X *and* B *a regular open set of* X. *Then, we have*

(1) $A \cap B$ *is* S-*closed relative to* B.

(2) B *is* S-*closed relative to* X *if* $B \subset A$.

(3) $Int_X(A)$ *is* S-*closed relative to* X *if* A *is closed in* X.

Proof. These follow from Theorem 3.2 and Theorem 3.3.

COROLLARY 3.4. *If* A *is an* S-*closed open subspace of a space* X *and* B *is a regular open set of* X, *then* $A \cap B$ *is an* S-*closed subspace of* X (*hence* A *and* B).

Proof. This follows from Theorem 3.1, Theorem 3.3 and Corollary 3.1.

THEOREM 3.4. *If* A *is* S-*closed relative to a space* X, *then* $Cl_X(A)$ *and* $Int_X(Cl_X(A))$ *are* S-*closed relative to* X.

Proof. Let $\mathscr{V} = \{V_\alpha \mid \alpha \in \nabla\}$ be a cover of $Cl_X(A)$ and $V_\alpha \in SO(X)$ for each $\alpha \in \nabla$. Then, $\mathscr{V}$ is a cover of A and A is S-closed relative to X. Therefore, there exists a finite subfamily ∇_0 of ∇ such that $A \subset \cup\{Cl_X(V_\alpha) \mid \alpha \in \nabla_0\}$. Thus we have $Cl_X(A) \subset \cup\{Cl_X(V_\alpha) \mid \alpha \in \nabla_0\}$. This shows that $Cl_X(A)$ is S-closed relative to X. Moreover, $Int_X(Cl_X(A))$ is regular open in X and hence, by Theorem 3.3 $Int_X(Cl_X(A))$ is S-closed relative to X.

COROLLARY 3.5. *If* A *is* S-*closed relative to a space* X, *then* $\mathrm{Int}_X(\mathrm{Cl}_X(A))$ *is an* S-*closed subspace of* X.

Proof. This follows from Theorem 3.1 and Theorem 3.4.

THEOREM 3.5. *If* A *is an* S-*closed open subspace of a space* X, *then* $\mathrm{Cl}_X(A)$ *is an* S-*closed subspace of* X.

Proof. Let $\mathscr{V} = \{V_\alpha \mid \alpha \in \nabla\}$ be a cover of $\mathrm{Cl}_X(A)$ and $V_\alpha \in \mathrm{SO}(\mathrm{Cl}_X(A))$ for each $\alpha \in \nabla$. Since A is open in X, we have $\mathrm{Cl}_X(A) \in \mathrm{SO}(X)$ and hence $V_\alpha \in \mathrm{SO}(X)$ for each $\alpha \in \nabla$ [3, Theorem 1]. Since A is open and S-closed, by Theorem 3.1 A is S-closed relative to X and $\mathscr{V}$ is a cover of A. Therefore there exists a finite subfamily ∇_0 of ∇ such that $A \subset \cup \{\mathrm{Cl}_X(V_\alpha) \mid \alpha \in \nabla_0\}$. Thus, we have

$$\mathrm{Cl}_X(A) = \cup \{\mathrm{Cl}_X(V_\alpha) \cap \mathrm{Cl}_X(A) \mid \alpha \in \nabla_0\} = \cup \{\mathrm{Cl}_{\mathrm{Cl}_X(A)}(V_\alpha) \mid \alpha \in \nabla_0\}.$$

This shows that $\mathrm{Cl}_X(A)$ is an S-closed subspace of X.

THEOREM 3.6. *If* A_1 *and* A_2 *are sets* S-*closed relative to a space* X, *then* $A_1 \cup A_2$ *is* S-*closed relative to* X.

Proof. Let $\mathscr{V} = \{V_\alpha \mid \alpha \in \nabla\}$ be a cover of $A_1 \cup A_2$ and $V_\alpha \in \mathrm{SO}(X)$ for each $\alpha \in \nabla$. Then $\mathscr{V}$ is a semi-open cover of A_i for $i = 1, 2$. Therefore, there exists a finite subfamily ∇_i of ∇ such that $A_i \subset \cup \{\mathrm{Cl}_X(V_\alpha) \mid \alpha \in \nabla_i\}$. Thus, we have

$$A_1 \cup A_2 \subset \cup \{\mathrm{Cl}_X(V_\alpha) \mid \alpha \in \nabla_1 \cup \nabla_2\}.$$

This shows that $A_1 \cup A_2$ is S-closed relative to X.

THEOREM 3.7. *Let* X *be an* S-*closed space and* A *a closed set of* X. *If the frontier* Fr (A) *of* A *is* S-*closed relative to* X, *then* A *is* S-*closed relative to* X.

Proof. Since A is closed in X, $\mathrm{Int}_X(A)$ is regular open and hence by Theorem 3.3 it is S-closed relative to X. Therefore, by Theorem 3.6 $A = \mathrm{Int}_X(A) \cup \mathrm{Fr}(A)$ is S-closed relative to X.

4. LOCALLY S-CLOSED SPACES

DEFINITION 4.1. A space X is said to be *locally* S-*closed* if each point of X has an open neighborhood which is an S-closed subspace of X.

It is obvious that every S-closed space is locally S-closed space. However, the converse is not true as the following example shows.

Example 4.1. An infinite discrete space is locally S-closed but not S-closed.

The space (R , Γ) in Example 2.1 is S-closed and hence locally S-closed but not locally compact. Therefore, every S-closed space is not necessarily locally compact. However, since S-closed regular space is compact [4, Lemma 1.4], every locally S-closed regular space is locally compact.

THEOREM 4.1. *For a space* X *the following are equivalent*:

(1) X *is locally* S-*closed*.

(2) *Each point of* X *has an open neighborhood which is* S-*closed relative to* X.

(3) *Each point of* X *has an open neighborhood* V *such that* $\mathrm{Cl_X(V)}$ *is* S-*closed relative to* X.

(4) *Each point of* X *has an open neighborhood* V *such that* $\mathrm{Int_X(Cl_X(V))}$ *is* S-*closed relative to* X.

(5) *Each point of* X *has an open neighborhood* V *such that* $\mathrm{Int_X(Cl_X(V))}$ *is an* S-*closed subspace of* X.

Proof. It follows from Theorem 3.1 that (1) implies (2) and (4) implies (5). By Theorem 3.4, (2) implies (3). By Theorem 3.3, (3) implies (4). It is obvious that (5) implies (1).

THEOREM 4.2. *If* X *is a locally* S-*closed space and* A *is a regular open set of* X, *then* A *is locally* S-*closed*.

Proof. Since X is locally S-closed, by Theorem 4.1 for each $x \in \mathrm{A}$ there exists an open neighborhood V of x such that V is S-closed relative to X. Since A is regular open in X, by Corollary 3.3 $\mathrm{A \cap V}$ is S-closed relative to A. Since $\mathrm{V \cap A}$ is an open neighborhood of x in A, by Theorem 4.1 A is locally S-closed.

THEOREM 4.3. *A space* X *is locally* S-*closed if and only if for each* $x \in \mathrm{X}$ *there exists an open set* A *of* X *such that* $x \in \mathrm{A}$ *and* A *is locally* S-*closed*.

Proof. *Necessity*. The proof is obvious.

Sufficiency. Let x be any point of X. Then, there exists an open set A such that $x \in \mathrm{A}$ and A is locally S-closed. Thus, there exists an open neighborhood V of x in A which is an S-closed subspace of A. Since A is open in X, V is open in X and hence by Corollary 3.1 V is an S-closed subspace of X. This shows that X is locally S-closed.

THEOREM 4.4. *If* X *is a locally* S-*closed space and* $f : \mathrm{X} \to \mathrm{Y}$ *is an open and continuous surjection, then* Y *is locally* S-*closed*.

Proof. For each $y \in \mathrm{Y}$, there exists $x \in \mathrm{X}$ such that $f(x) = y$. Since X is locally S-closed, by Theorem 4.1 there exists an open neighborhood U of x which is S-closed relative to X. Since f is open and continuous, $f(\mathrm{U})$ is an open neighborhood of y and S-closed relative to Y [4, Theorem 2.1]. Therefore, by Theorem 4.1 Y is locally S-closed.

THEOREM 4.5. *Let* $\{\mathrm{X}_\alpha \mid \alpha \in \nabla\}$ *be any family of spaces. If the product space* $\prod_{\alpha \in \nabla} \mathrm{X}_\alpha$ *is locally* S-*closed, then* X_α *is locally* S-*closed for each* $\alpha \in \nabla$.

Proof. The natural projection is an open and continuous surjection. Therefore, this follows immediately from Theorem 4.4.

References

[1] N. Levine (1963) – *Semi-open sets and semi-continuity in topological spaces*, « Amer. Math. Monthly », *70*, 36–41.

[2] O. Njåstad (1965) – *On some classes of nearly open sets*, « Pacific J. Math. », *15*, 961–970.

[3] T. Noiri (1973) – *On semi-continuous mappings*, « Atti Accad. Naz. Lincei, Rend. Cl. Sci. fis. mat. nat. », (8) *54*, 210–214.

[4] T. Noiri – *On* S*-closed spaces*, « Ann. Soc. Sci. Bruxelles » (to appear).

[5] V. Pipitone and G. Russo (1975) – *Spazi semiconnessi e spazi semiaperti*, « Rend. Circ. Mat. Palermo », (2) *24*, 273–285.

[6] T. Thompson (1976) – S*-closed spaces*, « Proc. Amer. Math. Soc. », *60*, 335–338.