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Semi partial geometries in $PG(2, q)$ and $PG(3, q)$

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Geometria. — *Semi partial geometries in* PG $(2, q)$ *and* PG $(3, q)$. Nota di INGRID DEBROEY e JOSEPH A. THAS, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si determinano tutte le geometrie semiparziali immergibili negli spazi proiettivi PG $(2, s)$, PG $(3, s)$.

1. INTRODUCTION

A semi partial geometry [7] is an incidence structure $S = (P, B, I)$ for which the following properties are satisfied:

(i) any point is incident with $u + 1$ $(u \geq 1)$ lines and two distinct points are incident with at most one line;

(ii) any line is incident with $s + 1$ $(s \geq 1)$ points and two distinct lines are incident with at most one point;

(iii) if two points are not collinear, then there are α $(\alpha > 0)$ points collinear with both;

(iv) if a point x and a line L are not incident, then there are 0 or t $(t \geq 1)$ points x_i and respectively 0 or t lines L_i such that $x \mathrel{I} L_i \mathrel{I} x_i \mathrel{I} L$.

Let $|P| = v$ and $|B| = b$. Then $b(s+1) = v(u+1)$ and $v = 1 + (u+1)s(1 + u(s-t+1)/\alpha)$ [7]. Also $t^2 \leq \alpha \leq (u+1)t$, $D = (u(t-1) + s - 1 - \alpha)^2 + 4((u+1)s - \alpha)$ is a square, except for $u = s = t = \alpha = 1$, and $((u+1)s + (v-1)(u(t-1) + s - 1 - \alpha + \sqrt{D})/2)/\sqrt{D}$ is an integer [7].

If the points (resp. lines) x and y (resp. L and M) are collinear (resp. concurrent), then we write $x \sim y$ (resp. $L \sim M$); otherwise we write $x \nsim y$ (resp. $L \nsim M$).

If a point x and a line L are not incident, then $\langle x, L \rangle$ denotes the number of lines which are incident with x and concurrent with L.

A semi partial geometry is called a partial geometry iff for any point x and any line L, with $x \not\mathrel{I} L$, we have $\langle x, L \rangle = t$ [2]. Equivalently S is a partial geometry iff $\alpha = (u+1)t$.

A semi partial geometry (resp. a partial geometry) for which $t = 1$, is called a partial quadrangle [5] (resp. a generalized quadrangle).

(*) Nella seduta dell'11 febbraio 1978.

Recently all partial geometries with B a lineset of PG (n, q), $n \geq 2$, P the set of all points of PG (n, q) on these lines, and I the natural incidence relation, were determined (the case $t = 1$ was handled by F. Buekenhout and C. Lefèvre ([4], [11]), the case $t > 1$ by F. De Clerck and J. A. Thas [8]).

In this paper we determine all semi partial geometries with parameters u, s, t and α, which are embeddable in PG $(2, s)$ or PG $(3, s)$.

2. THEOREM. *If* $S = (P, B, I)$ *is a semi partial geometry with parameters* u, s, t *and* α, *which is embedded in the projective plane* PG $(2, s)$, *then* S *is a dual design.*

Proof. As in a projective plane any two lines are concurrent, it is clear that S is a partial geometry with parameters u, s and $t = u + 1$. So S is a dual design [3].

3. THEOREM. *If* $S = (P, B, I)$ *is a semi partial geometry with parameters* u, s, t *and* α, *which is embedded in the projective space* PG $(3, s)$, *but not in a* PG $(2, s)$, *then the following cases may occur*:

(a) S *is a classical generalized quadrangle*;

(b) S *is the design of points and lines of* PG $(3, s)$;

(c) P *is the set of points of* PG $(3, s)$ *which are not on a given line of* PG $(3, s)$, *and* B *is the set of lines of* PG $(3, s)$ *which have no point in common with that line*;

(d) P *is the set of all points of* PG $(3, s)$ *and* B *is the set of all lines which are not totally isotropic for a given symplectic polarity of* PG $(3, s)$;

(e) S *is a Desargues configuration in* PG $(3, 2)$.

Proof. If S is a partial geometry, then (a), (b) and (c) are the only cases which may occur ([4], [8]).

Now we suppose that S is not a partial geometry. Then $u \geq s$ [7] and $t^2 \leq \alpha \leq ut$ [7].

If $t = 1$, $v = 1 + (u + 1) s (1 + us/\alpha)$. So $v \leq s^3 + s^2 + s + 1$ implies $u (u + 1) s \leq \alpha (s^2 + s - u)$, and consequently $u (u + 1) s \leq u (s^2 + s - u)$ or $u \leq s - 1 + \dfrac{1}{s+1} < s$, *a* contradiction. So we may assume that $2 \leq t \leq s$ (if $t = s + 1$, S is a design and thus a partial geometry).

Let V be a plane of PG $(3, s)$. As in PG $(3, s)$ any line has at least one point in common with V, and as $B \neq \varnothing$, V has at least one point in common with P. Now let M_1^V be the set of all points of V which are on a line $L \in B$ contained in V. Further let $M_0^V = (V \cap P) - M_1^V$.

A priori there exist three types of planes in PG $(3, s)$. A plane V is said to be of type I iff $M_1^V = \varnothing$, of type II iff there exists a line $L \in B$ such that $M_1^V = \{x \in P \| x \mathrm{I} L\}$, and of type III iff there exists a line $L \in B$ and a point $z \in P$ which are not incident and such that $M_1^V \supset \{x \in P \| x \mathrm{I} L\} \cup \{z\}$.

Suppose that there exists a plane V of type I. Then $b = |M_0^V| (u + 1)$ and so $|M_0^V| = v/(s+1)$ (1). If x is a point of M_0^V, then the number of ordered pairs (y, z) for which $x \sim y$, $y \sim z$ and $z \in M_0^V$ is given by $(|M_0^V| - 1)\alpha = (u+1)su$, and so $|M_0^V| = 1 + (u+1)su/\alpha$ (2). From (1) and (2) there follows that $\alpha = (u+1)t$, a contradiction. So any plane of PG $(3, s)$ is of type II or III.

Remark that there always exists a plane of type III (if $x \in P$ and if L and M are two lines of B incident with x, then the plane containing L and M is of type III). Now let V be a plane of type III. If L is a line of B in V, and if $x \in M_1^V$ is not incident with L, then there is at least one line M of B in V which is incident with x. As $M \sim L$, $\langle x, L\rangle = t$. Since any line of B in V is concurrent with L, there are exactly t lines of B in V which are incident with x. From $t \geq 2$ there follows that for any point $y \mathrm{I} L$ there is a line M of B in V which is not incident with y. Consequently any point of M_1^V is incident with exactly t lines of B in V.

Hence the incidence structure $S_V = (M_1^V, B_V, I_V)$, with $B_V = \{L \in B \| L$ is in $V\}$ and I_V induced by I, is a partial geometry with parameters $u' = t - 1$, $s' = s$ and $t' = t$. This implies $|M_1^V| = (s+1)(s+1-s/t)$. And so, as $b = |M_1^V|(u+1-t) + |M_1^V| t/(s+1) + |M_0^V|(u+1)$, we have $|M_0^V| = v/(s+1) - (s+1-st/(u+1))(s+1-s/t)$. Moreover, by considering the planes containing two non-collinear points of S, we see that α is divisible by t^2.

Now we suppose that all the planes of PG $(3, s)$ are of type III. Then considering a line L of B and all the planes through L, we get: $v = (s+1)(M_0 + M_1 - s - 1) + s + 1$, where $M_0 = v/(s+1) - (s+1-st/(u+1))(s+1-s/t)$ and $M_1 = (s+1)(s+1-s/t)$. This implies that $u = (t-1)(s+1)$ (3). On the other hand the number of ordered pairs (x, V), where $x \in P$ and V is a plane of PG $(3, s)$ containing x, is given by $v(s^2+s+1) = (s^3+s^2+s+1)(M_0+M_1)$. This together with (3) implies that $v = s^3 + s^2 + s + 1$, and so P is the set of all points of PG $(3, s)$. At last we remark that $v = 1 + (ts - s + t)s(1 + (t-1)(s+1)(s-t+1)/\alpha)$, and so $\alpha = (ts - s + t)(t-1)$. As $t^2 \mid \alpha$ and s is a prime power, there follows easily that $t = s$. Consequently $M_1 = s^2 + s$, $M_0 = 1$ and $S = (P, B, I)$ is a semi partial geometry with $u = s^2 - 1$, $s = t$ and $\alpha = s^2(s-1)$. We remark that for any plane V the structure S_V is a dual affine plane. Now it is immediately clear that $S' = (P, B', I')$, where $B' = \{$lines of PG $(3, s)\} \setminus B$ and where I' is the natural incidence relation, is a generalized quadrangle with parameters $u' = s$ and $s' = s$. From the theorem of Buekenhout-Lefèvre [4] there follows that S' is the generalized quadrangle W (s) arising from a symplectic polarity π [9] of PG $(3, s)$. In other words B is the set of all lines which are not totally isotropic [9] with respect to π.

Now we consider the case where there exists at least one plane V of type II. From $b = 1 + (s + 1)u + |M_0^V|(u + 1)$ there follows that $|M_0^V| = v/(s+1) - 1 - us/(u+1)$ (4). If x is a fixed point of M_1^V, then the number of ordered pairs $(y, z) \in P \times P$ such that $x \sim y$, $y \sim z$ and $z \in M_0^V$, is given by $us(u + 1 - t) = |M_0^V| \alpha$, and so $|M_0^V| = us(u + 1 - t)/\alpha > 0$ (5). Let x be a fixed point of M_0^V. Then the number of ordered pairs $(y, z) \in P \times P$ such that $x \sim y$, $y \sim z$ and $z \in V \cap P$, is given by $(u+1)su = (s + 1 + |M_0^V| - 1)\alpha$, and so $|M_0^V| = us(u+1)/\alpha - s$ (6). From (5) and (6) there follows that $\alpha = ut$, and from (4) and (5) there follows $u = s$. Consequently, $D = 1 + 4s(s + 1 - t)$ [7]. As $D \neq 5$, D must be a square [7], and so there exists an integer m such that $s(s + 1 - t) = m(m+1)$. As s is a prime power, s divides m or s divides $m + 1$. So we have $m \leq s + 1 - t < s \leq m + 1$ and thus $t = 2$ and $s = m + 1$. We conclude that $u = s$, $t = 2$, $\alpha = 2s$ and $D = (2s - 1)^2$. As $t^2 \mid \alpha$, $s = 2^h$ for some integer h. Finally $\sqrt{D} \mid (u+1)s + (v-1)(u(t-1) + s - 1 - \alpha + \sqrt{D})/2$ [7] implies that $u = s = t = 2$ and $\alpha = 4$ or $u = s = 8$, $t = 2$ and $\alpha = 16$. In the first case $S = (P, B, I)$ is a Desargues configuration in PG (3, 2).

Now we prove that the last case cannot occur. Suppose that $u = s = 8$, $t = 2$ and $\alpha = 16$. Then $|M_0^V| = 28$, for any plane V of type II. If W is a plane of type III, then $|M_0^W| = 0$, $|M_1^W| = 45$ and $|B_W| = 10$. Let V_1 be a plane of type II and let x and y be two points of $M_0^{V_1}$. Then as $\alpha > 0$, there is at least one plane V_2 of type III containing x and y. As $|M_0^{V_2}| = 0$ and S_{V_2} is a partial geometry with parameters $s' = s$, $u' = 1$, $t' = 2$, there holds $|V_2 \cap V_1 \cap P| = 5$. This implies that there are exactly 4 points of $M_0^{V_1}$ on the line joining x and y. So $M_0^{V_1}$ is a $\{28; 4\}$–arc, i.e. a maximal arc [1].

Now we consider a line $L \in B$. As $u = 8$ and as in any plane there are at most two lines of B incident with a given point, there are exactly 8 planes of type III and one plane of type II containing L. Now let x be a fixed point of P, and let $L_0, L_1, \cdots, L_8$ be the 9 lines of B incident with x. Through L_0 there is just one plane V_0 of type II. In this plane there is just one line R_0 different from L_0 and incident with x, which contains no point of $M_0^{V_0}$. So R_0 is a tangent to P, and consequently the plane containing R_0 and L_i, $i \in \{1, \cdots, 8\}$ is of type II. In other words the 9 planes $V_0, V_1, \cdots, V_8$ of type II corresponding with the 9 lines $L_0, L_1, \cdots, L_8$ all contain R_0.

Finally consider a plane V of type III. Let $N_0, N_1, \cdots, N_9$ be the 10 lines of B in V and let $V_0, V_1, \cdots, V_9$ be the 10 planes of type II corresponding with these lines. Suppose that 4 of these planes, say V_0, V_1, V_2, V_3, have a point x in common. If x_i is the point incident with N_0 and N_i, $i \in \{1, 2, 3\}$, then the line R_i joining x and x_i is a tangent of P. But then there are 3 lines R_1, R_2, R_3 in V_0 which are incident with x and which have no point in common with $M_0^{V_0}$, a contradiction. So $V_0, V_1, \cdots, V_9$ is a 10-arc [10] of the dual space of PG (3, 8). L. R. A. Casse however proved that for q even there is no $(q+2)$-arc in PG $(3, q)$ [6]. We conclude that the case $u = s = 8$, $t = 2$, $\alpha = 16$ cannot occur.

Remark. The geometric argument which shows that a semi partial geometry with parameters $u = s = 8$, $t = 2$ and $\alpha = 16$ cannot be embedded in PG (3, 8), holds for every semi partial geometry with parameters $u = s = 2^h$, $t = 2$ and $\alpha = 2s$.

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