Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Aldo Andreotti, Constantin Banica

Twisted sheaves on complex spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **64** (1978), n.2, p. 103–106.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1978_8_64_2_103_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1978.

RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta dell' 11 febbraio 1978

Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Twisted sheaves on complex spaces.* Nota di ALDO ANDREOTTI e CONSTANTIN BANICA, presentata (*) dal Corrisp. A. ANDREOTTI.

RIASSUNTO. — Sia X uno spazio complesso, $\mathscr{F}$ un fascio coerente su X, $\mathscr{L}$ un fascio localmente libero di rango uno, si studiano proprietà dei gruppi $H^r(X, \mathscr{F} \otimes \mathscr{L}^m)$ per m intero e $|m|$ molto grande.

Let $(X, \mathcal{O})$ be a complex space, $\mathscr{L}$ an invertible $\mathcal{O}$-module and $\mathscr{F}$ a coherent $\mathcal{O}$-module. Define $\mathscr{L}^m = \mathscr{L}^{\otimes m} = \mathscr{L} \otimes \cdots \otimes \mathscr{L}$ (m-time) for $m \geq 0$ ($\mathscr{L}^0 = \mathcal{O}$) and $\mathscr{L}^m = (\mathscr{L}^{-1})^{-m}$ for $m \leq 0$, where $\mathscr{L}^{-1}$ is the dual of $\mathscr{L}$. Now consider $\mathscr{F}(m) = \mathscr{F} \otimes \mathscr{L}^m$, the twisted sheaves of $\mathscr{F}$ relative to $\mathscr{L}$.

We present some results about these twisted sheaves.

a) We make the following agreement: if L is a holomorphic line bundle, then we will denote by $\mathscr{F}(m)$ the twisted sheaves of $\mathscr{F}$ relative to the sheaf of sections of L. Also, we shall write $m \gg 0$ to mean that the integer m is sufficiently large.

THEOREM 1. *Let* X *be a strongly q-pseudoconvex space of finite dimension,* $\mathscr{F}$ *an analytic coherent sheaf on* X *and* $\mathscr{F}(m)$ *the twisted sheaves associated to a positive holomorphic line bundle on* X. *Then*

(i) $H^r(X, \mathscr{F}(m)) = 0$ *for* $r > q$ *and* $m \gg 0$;

(ii) $H^r_k(X, \mathscr{F}(-m)) = 0$ *for* $r < \operatorname{prof} \mathscr{F} - q$ *and* $m \gg 0$.

(*) Nella seduta dell'11 febbraio 1978.

Statement (i) is nothing but Theorem 1 of [4]. For (ii) one uses (i), duality theory [9] and properties of dualizing sheaves [1], [2].

b) For a complex space X and $\mathscr{F} \in \mathrm{Coh}\, X$ we say that "$\mathscr{F}$ verifies the dual of Theorem A in dimension r" if the following is true:

for every x of X, any cohomology class $\xi \in H_x^r(X, \mathscr{F})$ which has a trivial image in $H_k^r(X, \mathscr{F})$, and such that $m_x \xi = 0$, is null (m_x is the maximal idela in x).

The following fact is useful: if $H_x^r(X, \mathscr{F}) \neq 0$, then there exists $\xi \in H_x^r(X, \mathscr{F})$, $\xi \neq 0$ and $m_x \xi = 0$.

Thus, if $\mathscr{F}$ verifies the dual of Theorem A in dimension r, and if for a point x the local cohomology $H_x^r(X, \mathscr{F})$ is $\neq 0$, then there exist cohomology classes in $H_k^r(X, \mathscr{F})$ with support $\{x\}$ (i.e. $\mathscr{F}$ has "enough cohomology classes with compact supports in dimension r").

THEOREM 2. *Let* X *be a strongly pseudoconvex space of finite dimension,* $\mathscr{F}$ *an analytic coherent sheaf on* X *and* $\mathscr{F}(m)$ *the twisted sheaves associated to a positive holomorphic line bundle on* X. *Then*

(i) $\mathscr{F}(m)$ *verifies Theorem* A *if* $m \gg 0$;

(ii) $\mathscr{F}(-m)$ *verifies the dual of Theorem* A *if* $m \gg 0$.

The proof of (i) uses point (i) of Theorem 1, Remmert's reduction, and the same argument used to prove that Theorem B of Cartan implies Theorem A; a little weaker assertion is proved in [6]. As for the proof of (ii) one uses duality for cohomology with supports a point.

THEOREM 3. *Under the hypothesis of Theorem* 2, *one has*:

(i) $\mathrm{prof}\, \mathscr{F} \geq q$ *if and only if* $H_k^r(X, \mathscr{F}(-m)) = 0$ *for* $r < q$ *and* $m \gg 0$;

(ii) $\dim \mathscr{F} \leq q$ *if and only if* $H_k^r(X, \mathscr{F}(-m)) = 0$ *for* $r > q$ *and* $m \gg 0$;

(iii) *for every* $m \gg 0$ *and any* $x \in X$ *such that* $\mathrm{prof}\, \mathscr{F}_x = q$ *or* $\dim \mathscr{F}_x = q$, *there exists a cohomology class* $\xi \in H_k^q(X, \mathscr{F}(-m))$ *with* $\mathrm{supp}\, \xi = \{x\}$.

For the proof, one needs the characterisation of "prof" and "dim" in terms of dualizing sheaves.

c) In the pseudoconcave case one has

THEOREM 4. *Let* X *be a strongly q-pseudoconcave space of finite dimension,* $\mathscr{F} \in \mathrm{coh}\, X$ *and* $\mathscr{F}(m)$ *the associated twisted sheaves corresponding to a negative holomorphic line bundle. Then*

(i) $H^r(X, \mathscr{F}(m)) = 0$ *for* $r < \mathrm{prof}(\mathscr{F}) - q - 1$ *and* $m \gg 0$;

(ii) $H_k^r(X, \mathscr{F}(-m)) = 0$ *for* $r > q + 1$ *and* $m \gg 0$ *if in addition* $\mathscr{F}$ *is Cohen-Macaulay.*

Statement (i) is nothing but Theorem 2 of [4]. For (ii) one need the following fact: if $\mathcal{F}$ is a Cohen-Macaulay sheaf on a complex space, then the dualizing sheaves $\mathcal{D}^q \mathcal{F}$ are null for $q \neq q_0 = \dim \mathcal{F}$ and $\mathcal{D}^{q_0} \mathcal{F}$ is also Cohen-Macaulay.

d) With the notations of the beginning, we denote also by

$$\mathcal{A}(X, \mathcal{L}) = \bigoplus_{m=0}^{\infty} \Gamma(X, \mathcal{L}^m) \quad \text{and} \quad \mathcal{M}^q(X, \mathcal{F}) = \bigoplus_{m=0}^{\infty} H^q(X, \mathcal{F}(m)) .$$

THEOREM 5. *Let* X *be a compact complex space*, $\mathcal{L}$ *an invertible sheaf on* X *and* $\mathcal{F} \in \operatorname{Coh} X$.

(i) *Assume* $\mathcal{A}(X, \mathcal{L})$ *without common zeros. Then the* **C**-*algebra* $\mathcal{A}(X, \mathcal{L})$ *is finitely generated and for every* q ; $\mathcal{M}^q(X, \mathcal{F})$ *is an* $\mathcal{A}(X, \mathcal{L})$-*module of finite type.*

(ii) *Assume* $\Gamma(X, \mathcal{L})$ *without common zeros. Then for every* q *the function* $m \to \dim H^q(X, \mathcal{F}(m))$ *is a polynomial od degree* $\leq \dim \mathcal{F}$ *for* $m \gg 0$ *and the function* $m \to \sum_q (-1)^q \dim H^q(X, \mathcal{F}(m))$ *is just a polynomial.*

(iii) *Assume that for any points* $x \neq x'$ *there exists a section* $s \in \Gamma(X, \mathcal{L})$ *such that* $s(x) = 0$ *and* $s(x') \neq 0$. *Then the degree of the polynomial* $m \to \Sigma(-1)^q \dim H^q(X, \mathcal{F}(m))$ *equals* $\dim \mathcal{F}$.

The theorem is in connection with some results of Zariski [11]. A particular case is proved in [3]. The proof uses the finitness theorem for graded sheaves of [5] and induction on $\dim \mathcal{F}$. One can also prove it using morphisms in some projective spaces defined by the global sections of $\mathcal{L}$ (when $\mathcal{L}$ has no fixed points), Grauert's coherence theorem [7], associated Leray's spectral sequences together with results of [10] (for the last assertion of (ii) one needs the invariance of Euler-Poincaré characteristic in a spectral sequence). The first argument works to more general situations, for instance one can prove the following statement:

" Let X be a compact complex space, $\mathcal{E}$ and $\mathcal{F}$ coherent analytic sheaves on X. Assume $\mathcal{E}$ generated by its global sections. Then the **C**-algebra $\mathcal{A}(X, \mathcal{L}) = \bigoplus_{m=0}^{\infty} \Gamma(X, S^m(\mathcal{E}))$ is finitely generated and for every q, $\bigoplus_{m=0}^{\infty} H^q(X, \mathcal{F} \oplus S^m(\mathcal{E}))$ is an $\mathcal{A}(X, \mathcal{L})$-module of finite type. In particular the functions $m \to \dim H^q(X, \mathcal{F} \oplus S^m(\mathcal{E}))$ are polynomial for $m \gg 0$ (here $S^m(\mathcal{E})$ denotes the m-th symmetric tensor power of $\mathcal{E}$) ".

e) Consider now twisted sheaves in the particular case $X = \mathbf{P}^n$ and $\mathcal{L} = \mathcal{O}_{\mathbf{P}^n}(1)$. In this case, a new question is when the freeness of a coherent sheaf can be deduced by means of its Hilbert polynomial.

THEOREM 6. *Let $\mathscr{F}$ be an analytic coherent sheaf on the projective space* $\mathbf{P}^n$ *and* $\mathscr{F}(m)$ *the twisted sheaves corresponding to the hyperplane section.*

(i) *Let us assume that the Hilbert polynomial* $m \to \chi(\mathbf{P}^n, \mathscr{F}(m))$ *equals the Hilbert polynomial of a free sheaf. Then* $\mathscr{F}$ *is free if and only if it is* 0*-regular; if in addition* $\mathscr{F}$ *is locally free, then these conditions are also equivalent with the fact that* $\mathscr{F}$ *is* (-1)*-coregular.*

(ii) *Assume* $\mathscr{F}$ *locally free; then* $\mathscr{F}$ *is a direct factor of a free sheaf* (*o finite rank*) *if and only if it is* 0*-regular and* (-1)*-coregular.*

Here the notion of m-regular for an integer m and a coherent sheaf $\mathscr{F}$ is that of ([8], lecture 11). By m-coregular we mean that

$$H^r(\mathbf{P}^n, \mathscr{F}(m-r)) = 0 \qquad \text{for} \quad r < \operatorname{prof} \mathscr{F} .$$

REFERENCES

[1] A. ANDREOTTI and C. BANICA (1975–1976) – *Relative duality on complex spaces*, « Revue Roum. de math. », *9*.

[2] A. ANDREOTTI and A. KAS (1973) – *Duality on complex spaces*, « Annali Sc. Norm. Sup. Pisa », *27*, 187–263.

[3] A. ANDREOTTI and G. TOMASSINI (1976) – *Some remarks on pseudoconcave manifolds*, *Essays in topology and related topics*, Mémoirs dédiés à G. de Rham, 85–104.

[4] A. ANDREOTTI and G. TOMASSINI (1969) – *A remark on the vanishing of certain cohomology groups*, « Compositio Mathematica », *21*, (4), 417–430.

[5] C. BANICA (1972) – *Le complété formel d'un espace analytique le long d'un sous-espace: un théorème de comparaison*, « Manuscripta math. », *6*, 207–244.

[6] S. ETO, H. KAZAMA and K. WATANABE (1974) – *On strongly q-pseudoconvex spaces with positive vector bundles*. « Memoirs of the Fac. of Sc., Kyushu University », ser. A, *28*, (2).

[7] H. GRAUERT (1960) – *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen*, « Publ. IHES », *5*.

[8] D. MUMFORD (1966) – *Lectures on curves on an algebraic surface*, « Annals of Mathematics Studies », Number 59, Princeton.

[9] J. P. RAMIS and G. RUGET, (1971) – *Complexe dualisant et théorèmes de dualité en Géométrie Analytique Complexe*, « Publ. Math. I.H.E.S. », *38*, 77–91.

[10] J. P. SERRE (1955) – *Faisceaux algébriques cohérents*, « Ann. of Math. », *61*, 197–278.

[11] O. ZARISKI (1962) – *The Theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface*, « Annals of Math. », *76* (3).