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A unilateral problem for a non linear vibrating string equation

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Analisi matematica. — *A unilateral problem for a non linear vibrating string equation.* Nota (*) del Corrisp. LUIGI AMERIO (**).

RIASSUNTO. — Si studia il moto di una corda vibrante, sotto l'azione di una forza esterna funzione dell'ascissa, del tempo e dello spostamento, nell'ipotesi che la corda sia vincolata a vibrare tra due ostacoli puntiformi, $G_1 = (\lambda(t), \alpha(t))$ e $G_2 = (\lambda(t), \beta(t))$, mobili, nel piano (x, y), con legge largamente arbitraria. La soluzione viene ricondotta a quella di un *problema elementare*, che si risolve col metodo delle approssimazioni successive.

1. INTRODUCTION

Consider the following, *non-linear, vibrating string equation*, in the *characteristic form* (and in the sense of *distributions*):

$$y_{\xi\eta} = f(\xi, \eta, y) = f(P, y), \tag{1.1}$$

where $\xi = (x + t)\,2^{-\frac{1}{2}}$, $\eta = (-x + t)\,2^{-\frac{1}{2}}$. In (1.1) $2f(P, y)$ denotes the external force, $y = y(P)$ is the displacement from the x axis, $t \geq 0$ is the time. We assume that the string, at rest, is placed on the x axis.

The aim of the present paper is to generalise to (1.1) the results recently obtained [1] when the external force does not depend on the displacement. We consider, as in [1], the following *unilateral problem*. Assume that the free vibration of the string, in the (x, y) plane, is impeded, from below and from above, by *a pair of point-shaped obstacles*, $G_1 = (\lambda(t), \alpha(t))$ and $G_2 = (\lambda(t), \beta(t))$, arbitrarily moving and through which the string is obliged to pass. We assume that $\alpha(t), \beta(t) \in C^0(0^{\vdash} + \infty)$ and that $\lambda(t)$ satisfies only the Lipschitz condition $|\lambda'(t)| \leq 1$ a.e., never being $\lambda'(t) = \pm 1$ on an interval: therefore the longitudinal velocity of the obstacles cannot be greater than the velocity of a wave traveling in the string, and the equality does not hold on an interval. We can also treat the simpler case of *one point-shaped obstacle* (that is $\alpha(t) \equiv -\infty$, or $\beta(t) = +\infty$) [2].

The problem considered has the following *analytical interpretation*. We consider, in the (x, t) plane, a line Λ, $x = \lambda(t)$, and we impose that the displacement $y(x, t)$ satisfies the following *pair of unilateral conditions*:

$$\alpha(t) \leq y(\lambda(t), t) \leq \beta(t) \qquad (t \geq 0). \tag{1.2}$$

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Let us observe that, as it is classical in *Mechanics*, such a problem can be reduced to a free problem by introducing the *reaction of the obstacle*, which is an *unknown distribution* J. The displacement $y(x, t)$ will therefore satisfy, *on the interior* $\mathring{Z}$ *of its domain* Z *of existence* (see fig. 3), *the equation*

$$y_{\xi\eta} = f(P, y) + J. \tag{1.3}$$

Assume now that the displacement $y(P) \in C^0(Z)$ and that, correspondingly, $f(P, y(P)) \in L^1_{loc}(Z)$ (this occurs, for instance, if conditions d), at § 4, hold). Denote moreover by $z(P)$ the solution, $\in C^0(Z)$, of the linear equation

$$z_{\xi\eta} = f(P, y(P)) \tag{1.4}$$

which has the *same* initial and boundary values as $y(P)$. Then, by (1.3) and (1.4),

$$J = \frac{\partial^2}{\partial\xi\,\partial\eta}(y(P) - z(P)) = \Gamma_{\xi\eta} \qquad (\Gamma(P) = y(P) - z(P)), \tag{1.5}$$

that is *the reaction of the obstacle coincides with the mixed second derivative of a function* $\Gamma(P) \in C^0(Z)$, with *null* initial and boundary values. By (1.3), (1.4) and (1.5), *the unknown functions*, $z(P)$ *and* $\Gamma(P)$, *satisfy the equation*

$$z_{\xi\eta} = f(P, z(P) + \Gamma(P)). \tag{1.6}$$

Moreover, the nature of the problem *imposes that*:

$$\operatorname{supp} J = \operatorname{supp} \Gamma_{\xi\eta} \subseteq \Lambda. \tag{1.7}$$

Therefore the function $\Gamma(P)$ satisfies the *homogeneous vibrating string equation*

$$\Gamma_{\xi\eta} = 0 \qquad \textit{on the whole of} \quad \mathring{Z} - \Lambda. \tag{1.8}$$

Other conditions (all of clear physical meaning) have to be added to (1.6) and (1.7) in order to prove *the existence and the uniqueness of the functions* $z(P)$ *and* $\Gamma(P)$ (cfr. § 4).

It is essential, to this aim, to solve an *elementary problem*, which we shall call $\pi_{f\alpha\beta}$ *problem* and which generalizes $\pi_{\alpha\beta}$ *problem* solved in [1].

2. PROPERTIES OF THE SOLUTION OF $\pi_{\alpha\beta}$ PROBLEM

Let us recall, firstly, the definition of $\pi_{\alpha\beta}$ *problem*.

Let us consider, in the (ξ, η) plane, the rectangle $R = oLNH = \{0 \le \xi \le l, 0 \le \eta \le h\}$, and let Λ be a line of equation

$$\eta = g(\xi) \qquad (0 \le \xi \le l), \tag{2.1}$$

where $g(\xi)$ is a *continuous*, *strictly increasing function*, $g(0) = 0, g(l) = h$.

Let moreover $\alpha(P)$, $\beta(P)$ be two *continuous* functions defined on Λ and such that

$$(2.2)\qquad \alpha(P) \leq \beta(P) \quad , \quad \alpha(0) \leq 0 \leq \beta(0).$$

Then $\pi_{\alpha\beta}$ *problem* consists in *finding a function* $\Gamma(P)$, $P \in R$, *which satisfies the following conditions* (where $\sigma = 0L \cup 0H$):

1) $\Gamma(P) \in C^0(R)$,

2) $\Gamma(P)|_\sigma = 0$ (*homogeneous Darboux forward condition*),

3) $\alpha(P) \leq \Gamma(P) \leq \beta(P) \quad \forall P \in \Lambda$,

4) $\operatorname{supp} \Gamma_{\xi\eta} \subseteq \{P \in \Lambda : \Gamma(P) = \alpha(P)$ *or* $\Gamma(P) = \beta(P)\}$,

5) $\Gamma_{\xi\eta} \geq 0$ *on every arc* $\Lambda' \subseteq \mathring{\Lambda}$ *where* $\Gamma(P) < \beta(P)$, $\Gamma_{\xi\eta} \leq 0$ *on every arc* $\Lambda'' \subseteq \mathring{\Lambda}$ *where* $\Gamma(P) > \alpha(P)$.

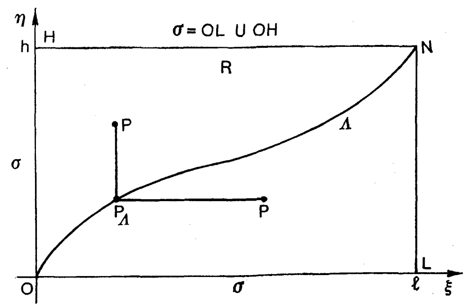

Fig. 1.

It is obvious, by 4), that $\Gamma_{\xi\eta}$ satisfies (1.8) on the open set

$$\mathring{R} - \operatorname{supp} \Gamma_{\xi\eta} \supseteq \mathring{R} - \Lambda .$$

It has been proved, in [1], that $\pi_{\alpha\beta}$ *problem admits one and only one solution*, $\Gamma(P) = \Gamma_{\alpha\beta}(P)$. Denoting moreover by $\Omega_{\alpha\beta}(P) = \Gamma_{\alpha\beta}(P)|_\Lambda$ the *trace* of $\Gamma_{\alpha\beta}(P)$ on Λ, *we have* (by 2.19 of [1])

$$(2.3)\qquad \Gamma_{\alpha\beta}(P) = \Omega_{\alpha\beta}(P_\Lambda) \qquad \forall P \in R$$

where P_Λ *is, on* Λ, *the maximum point* $\leq P$ (fig. 1).

Let us now prove the following *properties* of $\Gamma_{\alpha\beta}(P)$.

I) *Let* $\Gamma(P) = \Gamma_{\alpha\beta}(P)$ *be the solution of* $\pi_{\alpha\beta}$ *problem and let* $(\alpha_1(P), \beta_1(P))$ *be a pair of functions such that*

$$(2.4)\qquad \alpha(P) \leq \alpha_1(P) \leq \Gamma(P) \leq \beta_1(P) \leq \beta(P) \qquad \forall P \in \Lambda .$$

Then $\Gamma(P)$ *coincides with the solution* $\Gamma_1(P)$ *of* $\pi_{\alpha_1\beta_1}$ *problem*:

$$\Gamma(P) = \Gamma_{\alpha\beta}(P) = \Gamma_{\alpha_1\beta_1}(P) = \Gamma_1(P) \; (\Rightarrow \Gamma_{\alpha\beta}(P) = \Gamma_{\Gamma\beta}(P)) \,. \tag{2.5}$$

It is obvious that $\Gamma(P)$ satisfies the two first conditions characterizing the solution $\Gamma_1(P)$:

$1_1)$ $\Gamma(P) \in C^0(R)$, $\quad 2_1)$ $\Gamma(P)|_\sigma = 0$.

It follows moreover from (2.4):

$3_1)$ $\alpha_1(P) \leq \Gamma(P) \leq \beta_1(P) \qquad \forall P \in \Lambda$.

Assume now that $P_0 \in \operatorname{supp} \Gamma_{\xi\eta}$. Then we have $\Gamma(P_0) = \alpha(P_0)$, or $\Gamma(P_0) = \beta(P_0)$: assume $\Gamma(P_0) = \alpha(P_0)$.

It follows $\Gamma(P_0) = \alpha_1(P_0)$, as $\alpha(P_0) \leq \alpha_1(P_0) \leq \Gamma(P_0)$.

This implies the inclusion:

$$\{P \in \Lambda : \Gamma(P) = \alpha(P) \text{ or } \Gamma(P) = \beta(P)\} \subseteq$$
$$\subseteq \{P \in \Lambda : \Gamma(P) = \alpha_1(P) \text{ or } \Gamma(P) = \beta_1(P)\} \,.$$

Therefore $\Gamma(P)$ satisfies the condition:

$4_1)$ $\operatorname{supp} \Gamma_{\xi\eta} \subseteq \{P \in \Lambda : \Gamma(P) = \alpha_1(P) \text{ or } \Gamma(P) = \beta_1(P)\}$.

Assume lastly that it is $\Gamma(P) < \beta_1(P)$ on an arc $\Lambda_1' \subset \mathring{\Lambda}$; it follows $\Gamma(P) < \beta(P)$ on Λ_1'. Hence we have by 5):

$5_1)$ $\Gamma_{\xi\eta} \geq 0$ *on every arc* $\Lambda_1' \subset \mathring{\Lambda}$ *where* $\Gamma(P) < \beta_1(P)$,

$\Gamma_{\xi\eta} \leq 0$ *on every arc* $\Lambda_1'' \subset \Lambda$ *where* $\Gamma(P) > \alpha_1(P)$ and the thesis is proved.

II) (*Theorem of monotonic dependence*). *Let* $\Gamma(P)$ *and* $\Gamma_1(P)$ *be the solutions of* $\pi_{\alpha\beta}$ *and* $\pi_{\alpha\beta_1}$ *problems. Then, if it is*

$$\beta_1(P) \geq \beta(P) \qquad \forall P \in \Lambda \,, \tag{2.6}$$

we have also

$$\Gamma_1(P) \geq \Gamma(P) \qquad \forall P \in \Lambda \,. \tag{2.7}$$

If $\Gamma_1(P)$ *denotes the solution of* $\pi_{\alpha_1\beta}$ *problem, then*

$$\alpha_1(P) \leq \alpha(P) \Rightarrow \Gamma_1(P) \leq \Gamma(P) \,. \tag{2.8}$$

Assume that (2.6) holds and that there exists $C \in \Lambda$ such that $\Gamma_1(C) < \Gamma(C)$. As $\Gamma_1(0) = \Gamma(0) = 0$, there exists an arc $\overset{\frown}{AC} \subseteq \Lambda$ such that $\Gamma_1(P) < \Gamma(P)$ for $A < P \leq C$, and $\Gamma_1(A) = \Gamma(A)$. We have therefore

$\Gamma_1(P) < \beta(P) \leq \beta_1(P)$. This implies, by conditions 2) and 5), that the function $\Gamma_1(P)$ is increasing on $A^\frown C$ (cfr. (2.4) and (2.7) of [1]): therefore

$$\Gamma_1(P) \geq \Gamma_1(Q) \qquad \forall A \leq Q < P \leq C . \tag{2.9}$$

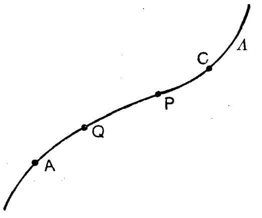

Fig. 2.

Moreover, since $\Gamma(P) > \Gamma_1(P) \geq \alpha(P)$, $\Gamma(P)$ is a decreasing function on $A^\frown C$:

$$\Gamma(P) \leq \Gamma(Q) \qquad \forall A \leq Q < P \leq C . \tag{2.10}$$

Hence, by (2.9) and (2.10):

$$\Gamma(C) \leq \Gamma(A) = \Gamma_1(A) \leq \Gamma_1(C) ,$$

which is absurd.

Let us observe that, by I and II, *the function* $\Gamma_1(P)$, *solution of* $\pi_{\alpha\beta_1}$ *problem, gives also the solution of* $\pi_{\Gamma\beta_1}$ *problem*: in fact

$$\alpha(P) \leq \Gamma(P) \leq \Gamma_1(P) \leq \beta_1(P) .$$

We shall denote, in what follows, by $\|\cdot\|$ and by $\|\cdot\|_\Lambda$ the maximum norm for the spaces $C^0(R)$ and $C^0(\Lambda)$. It is then obvious, by (2.3), that *the solution* $\Gamma_{\alpha\beta}(P)$ *of* $\pi_{\alpha\beta}$ *and its trace on* Λ, $\Omega_{\alpha\beta}(P)$, *have the same norm*:

$$\|\Gamma_{\alpha\beta}\| = \|\Omega_{\alpha\beta}\|_\Lambda = \|\Gamma_{\alpha\beta}\|_\Lambda . \tag{2.11}$$

III) *Let* $\Gamma(P)$ *and* $\Gamma_1(P)$ *be the solutions of* $\pi_{\alpha\beta}$ *and* $\pi_{\alpha\beta_1}$ *problems. Then, if it is*

$$\beta(P) \leq \beta_1(P) \qquad \forall P \in \Lambda , \tag{2.12}$$

it is also

$$\|\Gamma_1 - \Gamma\| \leq \|\beta_1 - \beta\|_\Lambda . \tag{2.13}$$

Assume the contrary, that is (by (2.11))

$$\|\Gamma_1 - \Gamma\| = \|\Gamma_1 - \Gamma\|_\Lambda > \|\beta_1 - \beta\|_\Lambda . \tag{2.14}$$

Let K be the closed set of the points $P \in \Lambda$ such that $\Gamma_1(P) - \Gamma(P) = \| \Gamma_1 - \Gamma \|$. Let C be the minimum point of K: we have $\Gamma_1(C) - \Gamma(C) = \| \Gamma_1 - \Gamma \| > 0, \Rightarrow C > 0$ (since $\Gamma_1(0) = \Gamma(0) = 0$). It is moreover

$$(2.15) \qquad 0 \leq \Gamma_1(P) - \Gamma(P) < \Gamma_1(C) - \Gamma(C) \qquad \text{for} \quad 0 \leq P < C.$$

Let us denote by T and by T_β the closed sets $\subseteq \Lambda$ such that

$$\Gamma(P) = \Gamma_1(P) \text{ on } T \quad , \quad \Gamma(P) = \beta(P) \text{ on } T_\beta .$$

Obviously, $C \notin T$ (since $\Gamma_1(C) > \Gamma(C)$); moreover, $C \notin T_\beta$ (as $C \in T_\beta \Rightarrow 0 \leq \leq \Gamma_1(C) - \Gamma(C) = \Gamma_1(C) - \beta(C) \leq \beta_1(C) - \beta(C) \leq \| \beta_1 - \beta \|_\Lambda < \| \Gamma_1 - \Gamma \|$). We have therefore, necessarily,

$$\Gamma(C) < \beta(C) \qquad \text{and} \quad \Gamma_1(C) > \Gamma(C) .$$

There exists then an arc $A^\frown C \subseteq \Lambda$, wiht $0 \leq A < C$, such that

$$\Gamma(P) < \beta(P) \qquad \text{and} \quad \Gamma_1(P) > \Gamma(P) \qquad \forall P \in A^\frown C .$$

Therefore $\Gamma(P)$ is an increasing function, on $A^\frown C$; conversely the function $\Gamma_1(P)$ is decreasing (as, by (2.5), $\Gamma_1(P) = \Gamma_{\Gamma\beta_1}$). It follows, $\forall P \in A^\frown C$,

$$\Gamma_1(P) - \Gamma(P) \geq \Gamma_1(C) - \Gamma(C) ,$$

contrary to (2.15). Hence (2.14) is absurd.

One proves, in the same way, *with reference to the solutions* $\Gamma_1(P)$ *and* $\Gamma(P)$ *of* $\pi_{\alpha_1\beta}$ *and* $\pi_{\alpha\beta}$ *problems, that*

$$(2.16) \qquad \alpha_1(P) \leq \alpha(P) \Rightarrow \| \Gamma_1 - \Gamma \| \leq \| \alpha_1 - \alpha \|_\Lambda .$$

We prove now the conclusive statement.

IV) (*Theorem of Lipschitz-continuous dependence*). *Let* $\Gamma(P)$ *and* $\Gamma_1(P)$ *be the solutions of* $\pi_{\alpha\beta}$ *and* $\pi_{\alpha_1\beta_1}$ *problems, with arbitrary pairs* (α, β) *and* (α_1, β_1).

We have then

$$(2.17) \qquad \| \Gamma_1 - \Gamma \| \leq 2 \{ \| \alpha_1 - \alpha \|_\Lambda + \| \beta_1 - \beta \|_\Lambda \} .$$

Setting, on the whole of Λ,

$$(2.18) \qquad \tilde{\alpha}(P) = \min \{ \alpha(P), \alpha_1(P) \} \quad , \quad \tilde{\beta}(P) = \max \{ \beta(P), \beta_1(P) \} .$$

we can consider, toghether with $\pi_{\alpha\beta}$ and $\pi_{\alpha_1\beta_1}$, the problems $\pi_{\alpha\tilde{\beta}}$, $\pi_{\tilde{\alpha}\tilde{\beta}}$, $\pi_{\tilde{\alpha},\beta_1}$ and the corresponding solutions. It follows from (2.18)

$$\tilde{\alpha}(P) \leq \alpha(P), \alpha_1(P) \quad ; \quad \tilde{\beta}(P) \geq \beta(P), \beta_1(P) .$$

Hence, by (2.12), (2.13), (2.16) and by (2.18),

$$\begin{aligned}\| \Gamma - \Gamma_1 \| &= \| \Gamma_{\alpha\beta} - \Gamma_{\alpha_1\beta_1} \| \leq \\ &\leq \| \Gamma_{\alpha\beta} - \Gamma_{\alpha\tilde{\beta}} \| + \| \Gamma_{\alpha\tilde{\beta}} - \Gamma_{\tilde{\alpha}\tilde{\beta}} \| + \| \Gamma_{\tilde{\alpha}\tilde{\beta}} - \Gamma_{\tilde{\alpha}\beta_1} \| + \| \Gamma_{\tilde{\alpha}\beta_1} - \Gamma_{\alpha_1\beta_1} \| \leq \\ &\leq \| \beta - \tilde{\beta} \|_\Lambda + \| \alpha - \tilde{\alpha} \|_\Lambda + \| \tilde{\beta} - \beta_1 \|_\Lambda + \| \tilde{\alpha} - \alpha_1 \|_\Lambda \leq \\ &\leq 2 \{ \| \alpha - \alpha_1 \|_\Lambda + \| \beta - \beta_1 \|_\Lambda \} ,\end{aligned}$$

which proves (2.17).

Let us observe that (2.17) *holds even with reference to other equivalent norms*; in particular *for the norms*

$$\| \Gamma \|' = \max_{R} | \vartheta (P) \Gamma (P) | \quad , \quad \| \Gamma \|'_\Lambda = \max_{\Lambda} | \vartheta (P) \Gamma (P) | ,$$

where $\vartheta (P)$, $P \in R$, *is a strictly positive, continuous and decreasing function.*

Let, in fact, $R_P = \{Q : 0 \leq Q \leq P\}$ be the rectangle $\subseteq R$ with maximum vertex P and let Λ_P be the part of Λ with ends 0 and P_Λ: we have, by (2.3) and (2.17),

$$\| \Gamma_1 - \Gamma \|_{R_P} \leq 2 \{ \| \alpha_1 - \alpha \|_{\Lambda_P} + \| \beta_1 - \beta \|_{\Lambda_P} \} .$$

Hence, $\forall P \in R$,

$$\begin{aligned}\vartheta (P) \, | \Gamma_1 (P) - \Gamma (P) | &\leq 2 \{ \vartheta (P) \| \alpha_1 - \alpha \|_{\Lambda_P} + \vartheta (P) \| \beta_1 - \beta \|_{\Lambda_P} \} \leq \\ &\leq 2 \{ \| \alpha_1 - \alpha \|'_{\Lambda_P} + \| \beta_1 - \beta \|'_{\Lambda_P} \} \leq 2 \{ \| \alpha_1 - \alpha \|'_\Lambda + \| \beta_1 - \beta \|'_\Lambda \} ,\end{aligned}$$

$$\| \Gamma_1 - \Gamma \|' \leq 2 \{ \| \alpha_1 - \alpha \|'_\Lambda + \| \beta_1 - \beta \|'_\Lambda \} . \tag{2.19}$$

Observation. – We can *generalise*, as in [1], $\pi_{\alpha\beta}$ problem, by substituting *homogeneous forward condition* 2) by an *arbitrary forward condition*:

2′) $$\Gamma (P) |_\sigma = \zeta_0 (P) ,$$

where $\zeta_0 (P) \in C^0 (\sigma)$ *and satisfies only the* (necessary) *inequalities*:

$$\alpha (0) \leq \zeta_0 (0) \leq \beta (0) . \tag{2.20}$$

Setting $P = \{\xi , \eta\}$, $P' = \{\xi , 0\}$, $P'' = \{0 , \eta\}$, the function

$$\zeta (P) = \zeta_0 (P') + \zeta_0 (P'') - \zeta_0 (0) \tag{2.21}$$

gives the solution of Darboux problem, for the homogeneous equation $\zeta_{\xi\eta} = 0$, with the *forward condition*:

$$\zeta (P) |_\sigma = \zeta_0 (P) . \tag{2.22}$$

Then *the generalised* $\pi_{\alpha\beta}$ *problem, with conditions* 1), 2′), 3), 4), 5), *has one and only one solution*:

$$\Gamma_{\alpha\beta} (P) = \zeta (P) + \overline{\Gamma} (P) . \tag{2.23}$$

In (2.23) $\overline{\Gamma}(\mathrm{P})$ *coincides with the solution* $\overline{\Gamma}_{\alpha-\zeta,\beta-\zeta}$ *of the* $\pi_{\alpha-\zeta,\beta-\zeta}$ *problem with homogeneous forward condition* (we calculate therefore $\overline{\Gamma}(\mathrm{P})$ by imposing 1), 2), $\cdots$, 5), where $\alpha(\mathrm{P})$ and $\beta(\mathrm{P})$ are substituted by $\alpha(\mathrm{P})-\zeta(\mathrm{P})$ and by $\beta(\mathrm{P})-\zeta(\mathrm{P})$).

Fix now the function $\zeta_0(\mathrm{P})$ *and consider two arbitrary pairs* (α_1,β_1), (α_2,β_2), *with*

$$\alpha_1(0), \alpha_2(0) \leq \zeta_0(0) \leq \beta_1(0), \beta_2(0).$$

It follow then from (2.19) and (2.23), *for the corresponding solutions* $\Gamma_{\alpha_1\beta_1}(\mathrm{P})$ *and* $\Gamma_{\alpha_2\beta_2}(\mathrm{P})$ (with $\Gamma_{\alpha_1\beta_1}|_\sigma = \Gamma_{\alpha_2\beta_2}|_\sigma = \zeta_0$):

$$\begin{aligned}\|\Gamma_{\alpha_1\beta_1}-\Gamma_{\alpha_2\beta_2}\|' = \|\overline{\Gamma}_{\alpha_1-\zeta,\beta_1-\zeta}-\overline{\Gamma}_{\alpha_2-\zeta,\beta_2-\zeta}\|' \leq \\ \leq 2\{\|\alpha_1-\alpha_2\|'_\Lambda + \|\beta_1-\beta_2\|'_\Lambda\}.\end{aligned} \tag{2.24}$$

3. $\pi_{f\alpha\beta}$ PROBLEM

Let $f(\xi,\eta,y)=f(\mathrm{P},y)$ be a function defined in the cylinder $\{\mathrm{P}\in\mathrm{R}, -\infty<y<+\infty\}$. *Assume that*

$$f(\mathrm{P},0)\in \mathrm{L}^1(\mathrm{R}) \tag{3.1}$$

and that $f(\mathrm{P},y)$ *is a Lipschitz-continuous function of* y, that is

$$|f(\mathrm{P},y_2)-f(\mathrm{P},y_1)| \leq \mathrm{K}\,|y_2-y_1|, \tag{3.2}$$

where the constant K does not depend on P, y_1, y_2.

Taken an *arbitrary* function $\zeta_0(\mathrm{P})\in \mathrm{C}^0(\sigma)$, let $\zeta(\mathrm{P})=\zeta_0(\mathrm{P}')+\zeta_0(\mathrm{P}'')-\zeta_0(0)$ be the solution of Darboux problem for the homogeneous equation $\zeta_{\xi\eta}=0$, with the forward condition $\zeta(\mathrm{P})|_\sigma = \zeta_0(\mathrm{P})$.

Let us consider now, on R, the *integral nonlinear equation* (of Volterra type):

$$y(\mathrm{P}) = \zeta(\mathrm{P}) + \int_{\mathrm{R_P}} f(\mathrm{Q}, y(\mathrm{Q}))\,d\mathrm{Q} \qquad (\mathrm{R_P}=\{0\leq \mathrm{Q}\leq \mathrm{P}\}). \tag{3.3}$$

As it is known, it follows from (3.1) and (3.2) that (3.3) *admits one and only one solution* $y(\mathrm{P})\in \mathrm{C}^0(\mathrm{R})$. Moreover, $y(\mathrm{P})$ *coincides with the solution* (in the sense of *distributions* $\in \mathrm{D}'(\mathring{\mathrm{R}})$) *of Darboux problem for the equation*

$$y_{\xi\eta} = f(\mathrm{P}, y), \tag{3.4}$$

satisfying the *forward condition*

$$y(\mathrm{P})|_\sigma = \zeta_0(\mathrm{P}). \tag{3.5}$$

We define now $\pi_{f\alpha\beta}$ *problem. Let us arbitrarily assign the functions*

$$z_0(P), \Gamma_0(P) \in C^0(\sigma) \qquad (3.6)$$
$$\alpha(P), \beta(P) \in C^0(\Lambda),$$

with

$$\alpha(0) \leq z_0(0) + \Gamma_0(0) \leq \beta(0). \qquad (3.7)$$

Then $\pi_{f\alpha\beta}$ *problem consists in determining a pair of functions,* $z(P)$ *and* $\Gamma(P)$, *such that:*

I) $z(P), \Gamma(P) \in C^0(R)$,

II) $z(P)|_\sigma = z_0(P) \quad , \quad \Gamma(P)|_\sigma = \Gamma_0(P)$,

III) $\alpha(P) \leq z(P) + \Gamma(P) \leq \beta(P) \qquad \forall P \in \Lambda$,

IV) $z_{\xi\eta} = f(P, z(P) + \Gamma(P))$ *on* $\mathring{R}$,

V) $\operatorname{supp} \Gamma_{\xi\eta} \subseteq \{P \in \Lambda : \Gamma(P) + z(P) = \alpha(P)$ *or* $\Gamma(P) + z(P) = \beta(P)\}$,

VI) $\Gamma_{\xi\eta} \geq 0$ *on every arc* $\Lambda' \subseteq \mathring{\Lambda}$ *where* $\Gamma(P) + z(P) < \beta(P)$,
$\Gamma_{\xi\eta} \leq 0$ *on every arc* $\Lambda' \subseteq \mathring{\Lambda}$ *where* $\Gamma(P) + z(P) > \alpha(P)$.

Let us observe that, by V), $\Gamma(P)$ *satisfies, on* $\mathring{R} - \Lambda$, *the equation* $\Gamma_{\xi\eta} = 0$, Setting then

$$y(P) = z(P) + \Gamma(P), \qquad (3.8)$$

it follows from IV) that $y(P)$ *satisfies, on* $\mathring{R} - \Lambda$, *the equation* $y_{\xi\eta} = f(P, y)$ (cfr. § 1).

We prove now that $\pi_{f\alpha\beta}$ *problem admits one and only one solution.* As we shall see, this can be proved by using classical Banach contraction theorem (hence, we can calculate the solution by successive approximations method).

Let us take, firstly, an arbitrary function $\Gamma(P) \in C^0(R)$ satisfying the condition

$$\Gamma(P)|_\sigma = \Gamma_0(P).$$

Setting afterwards $\zeta(P) = z_0(P') + z_0(P'') - z_0(0)$, we solve the integral equation

$$z(P) = \zeta(P) + \int_{R_P} f(Q, z(Q) + \Gamma(Q))\, dQ \qquad (P \in R) \qquad (3.9)$$

(which is possible, by (3.1) and (3.2)). Observe that the solution $z(P)$ is, $\forall$ fixed P, a functional of the restriction of Γ to the rectangle R_P; moreover

$$z(P)|_\sigma = z_0(P). \qquad (3.10)$$

Let lastly $\Delta(P)$ be the solution of the generalised $\pi_{\alpha-z,\beta-z}$ problem, with the condition

$$\Delta(P)\,|_\sigma = \Gamma_0(P)\,. \tag{3.11}$$

We have defined, in such a way, a functional transformation

$$\Delta = F(\Gamma)\,, \tag{3.12}$$

on the closed and convex set $\mathscr{U}$ consituted by all functions $\phi(P) \in C^0(R)$, such that

$$\phi(P)\,|_\sigma = \Gamma_0(P)\,.$$

Let us prove now that *the transformation* F *is a contraction, in* $\mathscr{U}$, *assuming* $C^0(R)$ *endowed with the norm*

$$\|\Phi\|' = \max_R |e^{-\rho(\xi+\eta)}\,\phi(\xi,\eta)|\,, \tag{3.13}$$

where

$$\rho = \text{const.} > \sqrt{5\,K} \qquad (\Longleftrightarrow 4\,K/(\rho^2 - K) < 1)\,. \tag{3.14}$$

Taken Γ_1 and $\Gamma_2 \in \mathscr{U}$, we have, by (3.2) and (3.9), for the corresponding solutions z_1 and z_2:

$$|z_1(P) - z_2(P)| \le K \int_{R_P} \{|z_1(Q) - z_2(Q)| + |\Gamma_1(Q) - \Gamma_2(Q)|\}\,dQ\,.$$

Setting $Q = (\mu, \nu)$, it follows from (3.13), $\forall P = (\xi, \eta) \in R$,

$$\begin{aligned} e^{-\rho(\xi+\eta)}\,|z_1(P) - z_2(P)| &\le K \int_{R_P} e^{-\rho(\xi-\mu)-\rho(\eta-\nu)}\,\{e^{-\rho(\mu+\nu)}\,|z_1(Q) - z_2(Q)| + \\ &\quad + e^{-\rho(\mu+\nu)}\,|\Gamma_1(Q) - \Gamma_2(Q)|\}\,d\mu\,d\nu \le \\ &\le \frac{K}{\rho^2}\,\{\|z_1 - z_2\|' + \|\Gamma_1 - \Gamma_2\|'\}\,, \\ &\Rightarrow \|z_1 - z_2\|' \le \frac{K}{\rho^2}\,\{\|z_1 - z_2\|' + \|\Gamma_1 - \Gamma_2\|'\}\,. \end{aligned}$$

Since, by (3.14), $K < \rho^2$, we obtain the inequality:

$$\|z_1 - z_2\|' \le \frac{K}{\rho^2 - K}\,\|\Gamma_1 - \Gamma_2\|', \tag{3.15}$$

where, by (3.14), $K/(\rho^2 - K) < 1$.

Let us calculate now the norm of $\Delta_1 - \Delta_2$. Setting

$$\alpha_1 = \alpha - z_1 \quad , \quad \beta_1 = \beta - z_1\,,$$
$$\alpha_2 = \alpha - z_2 \quad , \quad \beta_2 = \beta - z_2\,,$$

we have, by (2.24) and by (3.15),

$$\| \Delta_1 - \Delta_2 \|' \leq 4 \| z_1 - z_2 \|' \leq \frac{4\,K}{\rho^2 - K} \| \Gamma_1 - \Gamma_2 \|' \tag{3.16}$$

and the thesis follows from (3.14).

There exists therefore (by Banach theorem) *one and only one function* $\Gamma(P) \in \mathcal{U}$ *such that* $\Gamma = F(\Gamma)$. *The corresponding pair* $(z(P), \Gamma(P))$ *gives then the unique solution of* $\pi_{f\alpha\beta}$ *problem.*

4. Solution of the mechanical problem

We apply now the preceding results in order to solve the problem described at § 1. Let us consider equation (1.1) in a domain Z of the (x, t) plane, defined by the inequalities:

$$t \geq 0 \quad , \quad p(t) \leq x \leq q(t) , \tag{4.1}$$

where $p(t)$ and $q(t)$ satisfy Lipschitz conditions, and $p(t) < q(t)$, $\forall t$. We assume moreover $|p'(t)| \leq 1$, $|q'(t)| \leq 1$ a.e., never being $p'(t) = \pm 1$, or $q'(t) = \pm 1$, on an interval: therefore we exclude that the boundary lines, $\sigma_p = \{x = p(t)\}$ and $\sigma_q = \{x = q(t)\}$, contain any characteristic segment.

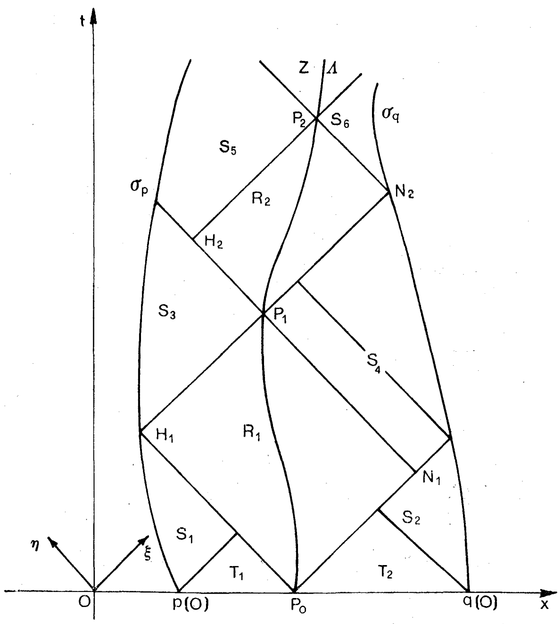

Fig. 3.

Suppose that there are assigned the *Cauchy initial conditions*:

$$y(x, 0) = \varphi(x) , y_t(x, 0) = \psi(x) , (p(0) \leq x \leq q(0)) , \tag{4.2}$$

and the *boundary conditions*:

$$(4.3) \qquad y(p(t), t) = A(t), y(q(t), t) = B(t) \qquad (t \geq 0).$$

Consider now a *line* $\Lambda \subset Z$, $x = \lambda(t)$, where $\lambda(t)$ satisfies the same type of conditions as $p(t)$ and $q(t)$ and it is

$$(4.4) \qquad p(t) < \lambda(t) < q(t) \qquad (t \geq 0).$$

The *data* are supposed to satisfy the following hypotheses:

a) $\varphi'(x), \psi(x) \in L^1(p(0) \vdash\!\dashv q(0))$,

b) $A(t), B(t) \in C^0(0 \vdash + \infty)$, $A(0) = \varphi(p(0))$, $B(0) = \varphi(q(0))$,

c) $\alpha(P), \beta(P) \in C^0(\Lambda)$, $\alpha(P_0) \leq \varphi(\lambda(0)) \leq \beta(P_0)$,

d) $f(P, 0) \in L^1(Z_T) \quad \forall T \geq 0$,

$$|f(P, y_2) - f(P, y_1)| \leq K_T |y_2 - y_1| \qquad \forall y_1, y_2 \text{ and } \forall P \in Z_T.$$

In *d*) $Z_T = \{0 \leq t \leq T, p(t) \leq x \leq q(t)\}$; K_T is a constant depending only on T. *Observe moreover that*, by *d*):

$$(4.5) \qquad \begin{aligned} & y_1(P), y_2(P) \in C^0(Z_T) \Rightarrow (f(P, y_1(P)) - f(P, y_2(P)) \in L^\infty(Z_T), \\ & y(P) \in C^0(Z_T) \Rightarrow f(P, y(P)) = f(P, 0) + (f(P, y(P)) - \\ & \qquad\qquad - f(P, 0)) \in L^1(Z_T). \end{aligned}$$

Let now W be *the set of all functions* $w(P)$ *such that*:

i_1) $w(P) \in C^0(Z)$,

i_2) $w_\xi(P), w_\eta(P), w_{\xi\eta}(P) \in L^1(T_1 \cup T_2)$.

In such hypotheses the *free problem* for (1.1), with the initial and boundary conditions (4.2) and (4.3), has one, and only one, solution $y(x, t) \in W$. We may obtain $y(x, t)$ by a classical scheme: we solve, firstly, a Cauchy problem in T_1 and in T_2; we solve, afterwards, Darboux and Goursat problems in $S_1, S_2, R_1, S_3, S_4, R_2, \cdots$. The solutions of these problems coincide with those of nonlinear integral equations (of Volterra type) for which existence and uniqueness are guaranteed by hypotheses *d*).

Let us consider now *the problem with obstacles*. Bearing in mind the description given at § 1, the initial and boundary conditions, and *setting*

$$(4.6) \qquad y(P) = z(P) + \Gamma(P),$$

we are brought to solve the following *analytical problem*: *find* $z(P)$ *and* $\Gamma(P)$ *such that*

j_1) $z(P)$ *and* $\Gamma(P) \in W$,

j_2) $z(x, 0) = \varphi(x) \quad , \quad z_t(x, 0) = \psi(x) \qquad (p(0) \leq x \leq q(0))$,

$$z(p(t), t) = \mathrm{A}(t) \quad , \quad z(q(t), t) = \mathrm{B}(t) \qquad (t \geq 0),$$
$$\Gamma(x, 0) = 0 \quad , \quad \Gamma_t(x, 0) = 0 \qquad (p(0) \leq x \leq q(0)),$$
$$\Gamma(p(t), t) = 0 \quad , \quad \Gamma(q(t), t) = 0 \qquad (t \geq 0),$$

$j_3)$ $\alpha(\mathrm{P}) \leq z(\mathrm{P}) + \Gamma(\mathrm{P}) \leq \beta(\mathrm{P}) \qquad \forall \mathrm{P} \in \Lambda,$

$j_4)$ $z_{\xi\eta} = f(\mathrm{P}, z(\mathrm{P}) + \Gamma(\mathrm{P}))$ *on* $\mathring{\mathrm{Z}}$,

$j_5)$ $\operatorname{supp} \Gamma_{\xi\eta} = \{\mathrm{P} \in \Lambda : \Gamma(\mathrm{P}) + z(\mathrm{P}) = \alpha(\mathrm{P})$ *or* $\Gamma(\mathrm{P}) + z(\mathrm{P}) = \beta(\mathrm{P})\}$,

$j_6)$ $\Gamma_{\xi\eta} \geq 0$ *on every arc* $\Lambda' \subset \mathring{\Lambda}$ *such that* $z(\mathrm{P}) + \Gamma(\mathrm{P}) < \beta(\mathrm{P})$,
$\Gamma_{\xi\eta} \leq 0$ *on every arc* $\Lambda'' \subset \mathring{\Lambda}$ *such that* $z(\mathrm{P}) + \Gamma(\mathrm{P}) > \alpha(\mathrm{P})$.

It is obvious that, if $\Gamma(\mathrm{P})$ and $z(\mathrm{P})$ satisfy $j_1), \cdots, j_6)$, then $y(\mathrm{P}) = z(\mathrm{P}) + \Gamma(\mathrm{P})$ gives a solution of our problem.

Let us prove now that *the pair* $(z(\mathrm{P}), \Gamma(\mathrm{P}))$ *exists and is unique.*

We have in fact, by the third and by the fourth of $j_2)$ (and solving the corresponding Cauchy and Goursat problems on $\mathrm{T}_1, \mathrm{T}_2$ and on $\mathrm{S}_1, \mathrm{S}_2$), $\Gamma(\mathrm{P}) \equiv 0$ on $\mathrm{T}_1 \cup \mathrm{T}_2 \cup \mathrm{S}_1 \cup \mathrm{S}_2$. We calculate now $z(\mathrm{P})$, on the same domain, by solving the same problems for the equation $z_{\xi\eta} = f(\mathrm{P}, z)$, where the initial and boundary values are given by the first and by the second of $j_2)$. Therefore, as it is obvious, the solution $y(\mathrm{P})$ and that of the free problem coincide on $\mathrm{T}_1 \cup \mathrm{T}_2 \cup \mathrm{S}_1 \cup \mathrm{S}_2$.

Observe now that $\Gamma(\mathrm{P})$ and $z(\mathrm{P})$ are known on the lower edges $\mathrm{P}_0 \mathrm{N}_1$ and $\mathrm{P}_0 \mathrm{H}_1$ of R_1. Hence we obtain $\Gamma(\mathrm{P})$ and $z(\mathrm{P})$ on the whole of R_1 by solving a $\pi_{f\alpha\beta}$ problem. We obtain then $\Gamma(\mathrm{P})$ on S_3 by solving a Goursat problem for the equation $\Gamma_{\xi\eta} = 0$; we can calculate afterwards $z(\mathrm{P})$, on S_3, by solving the same problem for the equation $z_{\xi\eta} = f(\mathrm{P}, z(\mathrm{P}) + \Gamma(\mathrm{P}))$. In the same way we obtain $\Gamma(\mathrm{P})$ and $z(\mathrm{P})$ on S_4, by solving Darboux and Goursat problems. We shall calculate then $\Gamma(\mathrm{P})$ and $z(\mathrm{P})$ in the rectangle R_2 by solving a $\pi_{f\xi\eta}$ problem and so on.

Our problem has been therefore solved. More generally, we can determine the motion of the string in presence of *more obstacles* of the type considered before. One assumes now that the domain Z contains *m lines* Λ_j, $x = \lambda_j(t)$, where $p(t) < \lambda_1(t) < \cdots < \lambda_m(t) < q(t)$; the displacement $y(\mathrm{P})$ must satisfy, correspondingly, *m conditions of the type*

$$\alpha_j(\mathrm{P}) \leq y(\mathrm{P}) \leq \beta_j(\mathrm{P}) \qquad (\mathrm{P} \in \Lambda_j).$$

It may be, for some j, $\alpha_j = -\infty$, *or* $\beta_j = +\infty$: in this, more simple, case we have *a point-shaped obstacle.*

For these problems, *existence and uniqueness theorem holds.*

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