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Unsteady Magnetoaerodynamic Forces on an Oscillating Circular Cylindrical Shell of Finite Length. Part III: Dynamic Response Problem

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Magnetofluidodinamica. — *Unsteady Magnetoaerodynamic Forces on an Oscillating Circular Cylindrical Shell of Finite Length.* Part III: *Dynamic Response Problem.* Nota di LIVIU LIBRESCU (*), presentata (**) dal Socio C. FERRARI.

RIASSUNTO. — Quest'ultima Nota, essendo la continuazione delle prime due [1, 2], contiene la deduzione delle equazioni che governano il problema della risposta dinamica e descrive un metodo analitico per risolverle.

1. In [2] it has been derived the expression of transient M—A forces acting on a finite elastic circular cylindrical shell flown by a supersonic, electrically-conducting gas, a magnetic field (with $\mathbf{H} \parallel \mathbf{U}$) being also present.

In order to determine the structural response of these panels subjected to an external pressure field, we start with the basic linearized equilibrium equations of the system

$$(\mathscr{S} - \mathscr{A} - \mathscr{I})(w) = \mathrm{F}^{\mathrm{E}}(x_1, x_2, t), \tag{1}$$

where F^{E} denotes the disturbing pressure field with prescribed spatial and temporal dependence, while $\mathscr{S}$, $\mathscr{I}$ and $\mathscr{A}$ denote respectively the structural, the inertial and the M—A operators, the last one being defined by Rel. (II.12_2) (1).

Adopting as valid the linear shallow theory of isotropic circular cylindrical shells, we have the following pertinent equation (see e.g. [3]).

$$(\mathscr{S} - \mathscr{I})(w) \equiv \mathrm{D}\Delta^2 w - \frac{1}{\mathrm{R}} \frac{\partial^2 \mathrm{C}}{\partial x_2^1} + m_0 \frac{\partial^2 w}{\partial t^2}, \tag{2}$$

$$(\Delta \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2),$$

where D and K denote the bending and stretching rigidities of the panel; m_0—the reduced mass; $\mathrm{C}(x_1, x_2, t)$—the potential function governed by

$$\mathrm{K}\Delta^2 \mathrm{C} = -\mathrm{R}^{-1} \partial^2 w/\partial x_1^2. \tag{3}$$

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(1) Prefix II is added to the relations and references afferent to Part II of the work (marked in the bibliography as [2]).

Considering simply-supported edge conditions and using Rels. (II.1) and (II.14), from (3), a particular solution is yielded as:

$$C(\bar{x}_1, \bar{x}_2, \bar{t}) = \frac{l^2}{R\pi^2} \sum_{q=1}^{N} \frac{q^2 V_q(\bar{t})}{\Delta_{q;\nu}} \sin q\pi\bar{x}_1 \cos n\theta, \tag{4}$$

where

$$\Delta_{q;\nu} = K(q^2+\nu^2)^2 \quad ; \quad \nu = nl/(\pi R).$$

Galerkin's method will now be used. For this purpose Rels. (II.1), (II.12), (II.14), (2) and (4) are inserted into Eq. (1) which is multiplied by $\sin r\pi\bar{x}_1 \times \times \cos n\theta$ and is further integrated over the panel mid-surface. Expressing F^E as $F^E = F(\bar{x}_1, \bar{x}_2) F_1(\bar{t})$, the following dimensionless set of N simultaneous integrodifferential equations, derived under zero initial conditions, is yielded as:

$$\begin{aligned}\sum_{q=1}^{N} \Bigg\{ \beta_1 \overset{(0)}{\Psi}_{rq} \ddot{v}_q(\bar{t}) + \Phi_{rq} v_q(\bar{t}) + \beta_2 \Bigg[c_1 \int_0^{\bar{t}} \mathscr{G}_{rq}(\bar{t}-\bar{\tau}) v_q(\bar{\tau}) d\bar{\tau} + \\ + c_1 \int_0^{\bar{t}} \overset{(2)}{\mathscr{F}}_{rq}(\bar{t}-\bar{\tau}) v_q(\bar{\tau}) d\bar{\tau} + 2M^2 \int_0^{\bar{t}} \overset{(1)}{\mathscr{F}}_{rq}(\bar{t}-\bar{\tau}) \dot{v}_q(\bar{\tau}) d\bar{\tau} + \\ + M^2 \int_0^{\bar{t}} \overset{(0)}{\mathscr{F}}_{rq}(\bar{t}-\bar{\tau}) \ddot{v}_q(\bar{\tau}) d\bar{\tau} \Bigg] \Bigg\} = C_r F_1(\bar{t}), \quad (\bar{t} \geq 0) \\ (r = 1, 2, \cdots, N), (\dot{v}_q(\bar{\tau}) \equiv dv_q/d\bar{\tau})\end{aligned} \tag{5}$$

where

$$\mathscr{G}_{rq}(\bar{t}-\bar{\tau}) = \int_0^1 W_q^{(1)}(\bar{x}_1)\Big|_{\bar{x}_1=0} W_r(\bar{x}_1) \tilde{K}(\bar{x}_1 ; \bar{t}-\bar{\tau}) d\bar{x}_1,$$

$$\overset{(m)}{\mathscr{F}}_{rq}(\bar{t}-\bar{\tau}) = \int_0^1 \tilde{K}(u ; \bar{t}-\bar{\tau}) \left[\int_u^1 W_r(\bar{x}_1) W_q^{(m)}(\bar{x}_1-u) d\bar{x}_1 \right] du,$$

are assumed locally integrable functions that are zero for $\bar{t}-\tau < 0$. Here the variable u is defined as $u \equiv \bar{x}_1 - \bar{\xi}_1$, while for the first moment of motion we have considered the instant $\bar{t}_0 = 0$.

In addition we have

$$\Phi_{rq} = \overset{(4)}{\Psi}_{rq} - 2\nu^2\pi^2 \overset{(2)}{\Psi}_{rq} + \nu^4\pi^4 \overset{(0)}{\Psi}_{rq} + \frac{q^4 \nu^4 R^2 \pi^4}{n^4 D \Delta_{q;\nu}} \overset{(0)}{\Psi}_{rq};$$

$$\overset{(m)}{\Psi}_{rq} = \int_0^1 W_r(\bar{x}_1) W_q^{(m)}(\bar{x}_1) d\bar{x}_1 \quad ; \quad \beta_1 = m_0 U^2 l^2 D^{-1};$$

$$\beta_2 = \mathscr{C}_1 a_0^2 \rho_0 l^3 D^{-1};$$

$$C_r = l^3 (\pi D)^{-1} \int_0^{2\pi} \int_0^1 F(\bar{x}_1, \bar{x}_2) W_r(\bar{x}_1) \cos n\theta \, d\bar{x}_1 d\theta,$$

while

$$W_q^{(m)}(\bar{x}_1) = d^m W_q / d\bar{x}_1^m \quad ; \quad W_q^{(0)}(\bar{x}_1) \equiv W_q(\bar{x}_1) = \sin q\pi\bar{x}_1 .$$

As for the circumferential wave number n, a particular value of it is to be selected for investigation.

The above derived set of integrodifferential equations governs the response problem of finite cylindrical panels placed in an electrically conducting gas flow and excited by an external pressure field; they depend not only upon the instantaneous characteristics of the system but also upon those of its past history.

A similar set of governing equations has been obtained by analysing the structural response problem of aerospace vehicles when considering the non-stationarity of the gas flow [4, 5].

2. In what follows an analytical framework for solving such equations of the convolution type will be presented. For this purpose the distributional point of view will be used.

In this connexion it is worth mentioning (see e.g. [II.3, II.4] that the space of distributions $\mathscr{D}'_+(\bar{t})$ whose supports are contained in $0 \leq \bar{t} < \infty$ constitute a commutative convolution algebra having the δ-distribution as unit element.

Let us proceed now to the extension by zero for $\bar{t} < 0$ of $F_1(\bar{t})$, $v_q(\bar{t})$, $\mathscr{G}_{rq}(\bar{t})$, $\cdots$ and to the utilization of some relations concerning the convolution process, i.e.

$$\text{(6)} \qquad T\delta(\bar{t}) * f(\bar{t}) = Tf \quad ; \quad Tf(\bar{t}) * g(\bar{t}) = f * Tg$$

where $*$ is the convolution symbol; T—an arbitrary differential operator having constant coefficients; f and g—distributions. Rels. (6) allow us to express Eqs. (5) as a set of N simultaneous convolution equations

$$\text{(7)} \qquad \sum_{q=1}^{N} \mathscr{C}_{rq}(\bar{t}) * v_q(\bar{t}) = \mathscr{B}_r(\bar{t}) , \qquad (r = 1, 2, \cdots, N)$$

where

$$\text{(8)} \qquad \mathscr{C}_{rq}(\bar{t}) = \beta_1 \overset{(0)}{\Psi}_{rq} \delta^{(2)}(\bar{t}) + \Phi_{rq}\,\delta(\bar{t}) + Y(\bar{t})\,\beta_1 [c_1 \mathscr{G}_{rq}(\bar{t}) + c_1 \overset{(2)}{\mathscr{F}}_{rq}(\bar{t}) +$$

$$+ 2M^2 \overset{(1)}{\mathscr{F}}{}^{(1)}_{rq}(\bar{t}) + M^2 \overset{(0)}{\mathscr{F}}{}^{(2)}_{rq}(\bar{t})] \quad , \quad (\delta^{(m)}(\bar{t}) \equiv d^m \delta / d\bar{t}^m \ ; \ \overset{(0)}{\mathscr{F}}{}^{(2)}_{rq}(\bar{t}) \equiv d^2 \overset{(0)}{\mathscr{F}}_{rq} / d\bar{t}^2)$$

and

$$\mathscr{B}_r(\bar{t}) = Y(\bar{t})\, C_r\, F_1(\bar{t}) , \qquad (r, q = 1, 2, \cdots, N),$$

are given distributions in $(\mathscr{D}'_+)_{N\times N}$ and $(\mathscr{D}'_+)_{N\times 1}$, respectively; $v_q(\bar{t})$ are the solutions also required to be in $(\mathscr{D}'_+)_{N\times 1}$, while $(\mathscr{D}'_+)_{N\times N}$ denotes the space of $N\times N$ matrices whose elements are in $\mathscr{D}'_+(\bar{t})$.

Now, we invoke a theorem in the field theory of convolution equations (see [II.3; II.4]), expressing the condition (necessary and suficient) for (7) to possess a unique solution in $(\mathscr{D}'_+)_{N\times 1}$. By virtue of this theorem the problem of finding a solution to (7) reverts to the problem of finding an inverse to the matrix $\mathscr{C} \equiv (\mathscr{C}_{rq})$ which in turn requires the existence of an inverse for det $\mathscr{C}$. If this is the case, there is precisely an unique solution in $(\mathscr{D}'_+)_{N\times 1}$ for v_q as expressed by

$$v_q(t) = \sum_{r=1}^{N} \tilde{\mathscr{C}}_{rq} * \mathscr{B}_r\,, \qquad (q = 1, 2, \cdots, N) \tag{9}$$

where $\tilde{\mathscr{C}} \equiv (\tilde{\mathscr{C}}_{rq})$ is the inverse of $\mathscr{C}$ in the convolution algebra $(\mathscr{D}'_+)_{N\times N}$, thus fulfilling (2) the relation $\tilde{\mathscr{C}} * \mathscr{C} = \delta_{N\times N}$ $\left(\text{or } \sum_{q=1}^{N} \tilde{\mathscr{C}}_{rq} * \mathscr{C}_{qs} = \delta_{(rs)} \text{ in indicial notations}\right)$, where $\delta_{(rs)}\,(\equiv \delta_{N\times N})$ denotes the $N\times N$ unit matrix in $\mathscr{D}'_+$ whose elements are:

$$\delta_{(rs)} = \delta \ \text{ for } \ r = s \ , \quad \delta_{(rs)} = 0 \ \text{ for } \ r \neq s\,, \qquad (r, s = 1, 2, \cdots, N).$$

3. In order to facilitate the determination of $\tilde{\mathscr{C}}$, we shall express most accurately the functions $\mathscr{G}_{rq}, \cdots, \overset{(0)}{\mathscr{F}}{}^{(2)}_{rq}$ occurring in $(8)_1$ through exponential polynomials, as under

$$Y(t)\,\mathscr{G}_{rq}(t) = Y(t) \sum_{\substack{l=1\\m=1}}^{L;M} g_{(rq)lm}\, t^{l-1}\, e^{\gamma_{(rq)m} t} \ ; \ \cdots, \ a.s.o. \qquad (r, q = 1, 2, \cdots, N) \tag{10}$$

where $q_{(rq)lm}\,, \gamma_{(rq)m} \in C$ (the class of complex constants) are distinct ones for each function to be approximated, while L and M are positive integers, their choice depending on the required accuracy of the approximation.

As it is well known (see [II.3]), the r.h.s. of (10) may be expressed as

$$Y(t) \sum_{\substack{l=1\\m=1}}^{L,M} g_{(rq)lm}\, t^{l-1}\, e^{\gamma_{(rq)m} t} = \sum_{\substack{l=1\\m=1}}^{L,M} g_{(rq)lm}\,(l-1)\,!\,(\delta^{(1)} - \gamma_{(rq)m}\,\delta)^{-l}. \tag{11}$$

By considering in (10) $L = 1$, which leads to (3)

$$Y(t)\,\mathscr{G}_{rq}(t) = Y(t) \sum_{m=1}^{M} g_{(rq)m}\, e^{\gamma_{(rq)m} t} \tag{12}$$

(2) Thus $(\tilde{\mathscr{C}}_{rq})$ plays the role of the matrix response of the system to an unit impulse.

(3) These functions, occurring in the governing equations of some analysed response problems [4, 5], are approximated *ab initio* as under (12) by considering in addition $M = 2$. For a general study of the approximation of functions through exponential polynomials and quasi-polynomials (as defined by Rels. (10) and (12), respectively) see [7], where an ample bibliography of the matter can be found.

one thus gets a considerable simplification of (11) consisting in

$$Y(t)\sum_{m=1}^{M} g_{(rq)m}\, e^{\gamma_{(rq)m} t} = \sum_{m=1}^{M} g_{(rq)m}\,(\delta^{(1)} - \gamma_{(rq)m}\,\delta)^{-1}. \tag{13}$$

Expressing in $(8)_1$ the distributions $Y(t)\mathscr{G}_{rq}(t), \cdots, Y(t)\overset{(0)}{\mathscr{F}}{}^{(2)}_{rq}(t)$ under the form given by the right-hand side of Rel. (11) and making use of Heaviside's symbolic method (consisting at this stage—see [6] and [II.3]—in the formal replacement of $\delta^{(n)}$ by p^n and of $\lambda\delta$ by λ, where λ denotes an arbitrary scalar), we are allowed to express $\mathscr{C}_{rq}(p)$ as:

$$\mathscr{C}_{rq}(p) = \frac{\sum_{n=0}^{m+2} {}_{(rq)}a_n\, p^n}{\sum_{n=0}^{m} {}_{(rq)}b_n\, p^n}, \qquad (r, q = 1, 2, \cdots, N) \tag{14}$$

where the positive integer m and the complex constants ${}_{(rq)}a_n$ and ${}_{(rq)}b_n$ depend upon the choice of r and q.

Having thus reduced the problem to a linear algebra one, we can obtain the matrix inverse of $\mathscr{C} \equiv (\mathscr{C}_{rq}(p))$ i.e. $\tilde{\mathscr{C}} \equiv (\tilde{\mathscr{C}}_{rq}(p))$ their elements being related through $\sum_{q=1}^{N} \tilde{\mathscr{C}}_{rq}(p)\,\mathscr{C}_{qs}(p) = 1_{(rs)}$, where $1_{(rs)}$ denotes the unit matrix whose elements are:

$$1_{(rs)} = 1 \quad \text{for} \quad r = s \quad ; \quad 1_{(rs)} = 0 \quad \text{for} \quad r \neq s.$$

If $\det \mathscr{C}(p) \neq 0$, then $\mathscr{C}(p)$ has an inverse given by $\tilde{\mathscr{C}}(p) = A(p)/(\det \mathscr{C}(p))$, where the adjoint $A(p)$ of $\mathscr{C}(p)$ is the matrix transpose of the matrix of the cofactors of $\mathscr{C}$.

Having obtained $\tilde{\mathscr{C}}_{rq}(p)$ under the form $\tilde{\mathscr{C}}_{rq}(p) = {}_{(rq)}Q_n(p)/{}_{(rq)}P_m(p)$ where P and Q are polynomials given by

$${}_{(rq)}Q_n(p) = \sum_{s=0}^{n} {}_{(rq)}\alpha_s\, p^s \quad ; \quad {}_{(rq)}P_m(p) = \sum_{s=0}^{m} {}_{(rq)}\beta_s\, p^s,$$

where α_s and $\beta_s \in C$, m and n being positive integers and supposing $n > m$, one easily obtains

$$\tilde{\mathscr{C}}_{rq}(p) = \sum_{\nu=0}^{n-m} {}_{(rq)}\xi_\nu\, p^\nu + \frac{\sum_{t=0}^{s} {}_{(rq)}\rho_t\, p^t}{\sum_{t=0}^{m} {}_{(rq)}\beta_t\, p^t}, \tag{15}$$

where $s < m$, ξ_ν and ρ_s being complex constants.

By carrying out a partial fraction expansion, from Rel. (15) we get

$$\tilde{\mathscr{C}}_{rq}(p) = \sum_{\nu=0}^{n-m} {}_{(rq)}\xi_\nu\, p^\nu + \sum_{\mu=1}^{g}\sum_{\nu=1}^{k_\mu} \frac{\zeta_{(rq)\mu\nu}}{(p - \gamma_{(rq)\mu})^\nu}, \quad \sum_{\mu=1}^{g} k_\mu = m; \tag{16}$$

$$(r, q = 1, 2, \cdots, N),$$

γ_μ being the complex roots of multiplicity k_μ of $P_m(p)$ and $\zeta_{\mu\nu}$ the coefficients of the expansion.

Reverting now to the initial symbol δ, i.e. replacing p^n by $\delta^{(n)}$ and λ by $\lambda\delta$ and having in view Rel. (11), from (15) we finally get

$$(17) \qquad \tilde{\mathscr{C}}_{rq}(\bar{t}) = \sum_{\nu=0}^{n-m} {}_{(rq)}\xi_\nu\, \delta^{(\nu)}(\bar{t}) + \sum_{\mu=1}^{g}\sum_{\nu=1}^{k_\mu} \frac{\zeta_{(rq)\mu\nu}}{(\nu-1)!}\, \bar{t}^{\nu-1} e^{\gamma_{(rq)\mu}\bar{t}}\, Y(\bar{t}) .$$

When $m > n$ the first summation term does not appear.

Further, by inserting (16) in (9) and by making use of Rel. (6) whenever possible, one obtains $v_q(\bar{t})$, wherefrom the transient deflection and stresses in the structure are determined by using Rel. (II.1) in conjunction with (II.14) and, respectively, the relations expressing the stresses in terms of w and C (see e.g. [3]).

Concerning the results obtained in Sect. 3, it is worth mentioning that these ones can also be deduced by using some alternative methods as e.g. the operational method developed in the space of $\mathscr{D}'_+$ distributions (see [6]) or the Laplace transform of right-sided distributions.

It is also worth remarking that the method exhibited in this Note may be used not only when dealing with the structural response problem to deterministic pressure excitations (as it has been done explicitely in this paper), but we may appropriately utilize Eq. (7)(4) to determine also the dynamic response when the system is subjected to a random pressure field.

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(4) For such an analysis, undertaken in another context, but starting with this type of equations, see e.g. [7].