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The Ehlers-Rindler problem in cylindrical symmetry

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Teorie relativistiche. — *The Ehlers-Rindler problem in cylindrical symmetry.* Nota I di MAURO CARFORA, presentata (*) dal Socio C. CATTANEO.

RIASSUNTO. — Si studia il campo gravitazionale e il campo elettromagnetico generati da uno strato cilindrico carico circondato da un secondo strato cilindrico coassiale neutro ed uniformemente rotante rispetto ad esso. Si trovano le soluzioni generali delle equazioni di Einstein–Maxwell nelle tre regioni separate dai due strati cilindrici, e si impongono poi le condizioni di raccordo. Le conclusioni fisiche sembrano in accordo con il punto di vista machiano. L'esposizione del lavoro viene suddivisa in due Note successive.

In 1969 [10], W. Rindler conjectured that a charged spherical shell, inside a neutral rotating one, would be surrounded, according to Mach's ideas, by a dipole-like magnetic field. This suggested to him and to J. Ehlers [5], [6] the statement of an interesting problem in general relativity, a problem that they solved to first order in the gravitational constant χ, and to second order in the angular velocity of the shell, ω; with results that, according to the Authors themselves, left some interpretative uncertainty.

The Ehlers-Rindler problem, (E-R problem henceforth), and the machian Thirring problem are similar to each other in many respects. It has been argued by E. Frehland [7] and L. Pietronero [9] that cylindrical symmetry, rather than the "almost-spherical" one adopted by Thirring himself and by many others, is relevant for a clear description of the Thirring effect. This seems to suggest that also the E-R problem finds a more natural collocation in the hypothesis of spatial cylindrical symmetry. Hence keeping untouched Rindler's original idea, we consider the following problem:

"to find the solution of the Einstein-Maxwell equations describing a space-time manifold V^4 and the electromagnetic fields generated by two coaxial and infinitely long cylindrical thin shells of matter. The inner shell is supposed to be at rest and charged, the outer shell, electrically neutral, is uniformly rotating round the common axis. Rest and motion being considered with respect to the static frame of reference outside the shells".

I wish to express my gratitude to Prof. Carlo Cattaneo for having suggested to me this problem and for his help in the course of this work.

Notation. Four-dimensional tensor indices are denoted by Latin letters $i, k, l, \cdots$, and take the values $1, 2, 3, 4$. Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, \cdots$. We use the metric with signature $+ + + -$.

(*) Nella seduta del 14 gennaio 1978.

1. Statement of the problem

Let us call $\overline{\Sigma}$ and Σ the hypersurfaces describing the histories of the inner and outer shell respectively. They divide V^4 in three regular regions: A_1, within the charged shell; A_2, between the shells; A_3, outside both. According to our hypotheses each region A_α is stationary and provided with spatial cylindrical symmetry: that is, it admits a group of isometries G_3, generated by three commuting Killing vectors $\xi_{(2)}, \xi_{(3)}, \xi_{(4)} \cdot \xi_{(2)}, \xi_{(3)}$, being space-like, $\xi_{(4)}$, time-like, and their trajectories being homeomorphic to S^1, R^1, R^1, respectively. Such hypotheses imply the existence, in each A_α, of physically admissible local coordinates (x^i), in which $g_{ik,2} = g_{ik,3} = g_{ik,4} = 0$, that we call stationary cylindrical coordinates of A_α. We have a continuous infinity of such coordinates, and one passes from a system to another by means of a transformation of the kind: $x^1 = f^1(x^{1'})$, $x^u = A^u_{u'} x^{u'} + f^u(x^{1'})$, $(u = 2, 3, 4)$, where, up to the invertibility conditions, $A^u_{u'}$, are arbitrary constants, and $f^1(x^{1'}), f^u(x^{1'})$ are arbitrary functions. Without any physical limitation, we can put $x^1 = x^{1'}$, $f^u(x^{1'}) = 0$, so we shall consider only stationary cylindrical coordinates systems defined up to:

$$x^1 = x^{1'} \quad , \quad x^u = A^u_{u'} x^{u'}. \tag{1}$$

Let us denote, in the regions A_1, A_2, A_3, such coordinates systems by

$$(x^{i'}) = (r', \varphi', z', ct'), (x^i) = (r, \varphi, z, ct), (x^{\hat{i}}) = (\hat{r}, \hat{\varphi}, \hat{z}, \hat{ct}),$$

respectively, and let us call $\zeta_{(u)}$, $(u = 2, 3, 4)$, the Killing vectors in the inner region A_1, $\xi_{(u)}$ those in the intermediate region A_2, $\vartheta_{(u)}$ those in the outer region A_3. The *congruences of the time-like trajectories of* $\zeta_{(4)}, \xi_{(4)}, \vartheta_{(4)}$, *realize the physical frames of reference, in Møller-Cattaneo's sense*, S_1, S_2, S_3, *respectively*. We assume that the coordinates $(x^{i'})$ are adapted to S_1, (x^i) to S_2, $(x^{\hat{i}})$ to S_3. In such coordinate systems, the hypersurfaces $\overline{\Sigma}$ and Σ are so characterized: $\overline{\Sigma}: r' = r_0$, with respect to $(x^{i'})$; $r = r_0$, with respect to (x^i); $\Sigma: r = R_0$, with respect to (x^i); $\hat{r} = R_0$, with respect to $(x^{\hat{i}})$. The points of $\overline{\Sigma}$ and Σ belong to $A_1 \cap A_2$, and to $A_2 \cap A_3$, respectively. Then, it follows that, on $\overline{\Sigma}$ the coordinates (x^i) must can be expressed as a function of the $(x^{i'})$. The same thing is true, on Σ, for the coordinates $(x^{\hat{i}})$ and (x^i). Such connections are the ones expressed by (1), that is, we must have:

$$x^u = A^u_{u'} x^{u'} \cdots \text{on } \overline{\Sigma} \quad , \quad x^{\hat{u}} = B^{\hat{u}}_u x^u \cdots \text{on } \Sigma, \tag{2}$$

$A^u_{u'}$, and $B^{\hat{u}}_u$ being, up to the invertibility conditions, arbitrary constants.

The local coordinates $(x^{i'}), (x^i), (x^{\hat{i}})$, *by definition, are comoving with the frames* S_1, S_2, S_3, *respectively. Therefore, in* (2), *the former relation describes on* $\overline{\Sigma}$, *the motion of the frame* S_1 *with respect to* S_2, *while the latter describes, on* Σ, *the motion of* S_2 *with respect to* S_3. *In our situation, corresponding to a relative rotation of the matter evolving on* $\overline{\Sigma}$ *and* Σ, *it seems reasonable to reduce* (2) *to the form*:

$$\varphi = \eta\varphi' + \lambda ct', z = z' \quad , \quad ct = \sigma\varphi' + \nu ct' \cdots \text{on } \overline{\Sigma}, \tag{3}$$

$$\hat{\varphi} = \alpha\varphi + \beta ct, \hat{z} = z \quad , \quad \hat{ct} = \gamma\varphi + \delta ct \cdots \text{on } \Sigma, \tag{3'}$$

describing the relative rotation between S_1, S_2 *and* S_2, S_3, *respectively.*

The constants $\omega(\rho, \mu)$:

$$\omega(1, 2) = c\lambda/\nu \quad , \quad \omega(2, 1) = -c\lambda/\eta \quad , \quad \omega(2, 3) = c\beta/\delta \quad , \quad \omega(3, 2) = -c\beta/\alpha$$

can be interpreted as the coordinate angular velocities of two contiguous frames, that is, of S_ρ *with respect to* S_μ. *Such constants, which will have an invariant characterization in terms of Killing vectors, will be determined requiring the junction conditions among the regions* A_α.

We shall assume that in each region A_α one can choose adapted coordinates which are time-orthogonal too: $g_{4\rho} = 0$ [4]. One can show that such a choice should not imply any loss of generality if the region A_α were empty. In the present situation such a condition does not hold since we are in the presence of e.m. fields, and the previous assumption must be considered as a useful simplifying hypothesis. Following such remarks we shall agree that the coordinates $(x^{i'}), (x^i), (x^{\hat{i}})$, are time-orthogonal in A_1, A_2, A_3, respectively, hence the physical frames of reference S_μ will be static in Levi-Civita's sense in the homonymous regions A_μ (and only there!).

Such frames seem to be the most natural ones in order to describe physics in our space-time.

According to our hypotheses we suppose that, with respect to S_3, *the charged shell is at rest, whilst the outer shell is uniformly rotating round the common axis, with a known standard angular velocity* ω (1).

2. THE EINSTEIN–MAXWELL EQUATIONS

In each region A_α, using adapted coordinates, we can write the metric in the general Levi-Civita's form [8], [2]:

$$ds^2 = e^{2(k-u)} (dr^2 + dz^2) + R^2(r)\, e^{-2u}\, d\varphi^2 - c^2 e^{2u}\, dt^2, \tag{4}$$

$k(r), u(r), R(r)$ being unknown functions, different for each region A_α, that we have to find by means of the gravitational equations $G^i_k = -\chi S^i_k$, G^i_k and S^i_k being the Einstein-Levi-Civita's tensor and the energy-momentum tensor of the e.m. field, respectively:

$$G^i_k \equiv R^i_k - \frac{1}{2}\delta^i_k R \quad , \quad S^i_k \equiv F_{kl} F^{il} - \frac{1}{4}\delta^i_k F_{lm} F^{lm}.$$

(1) With respect to a frame Ξ, represented by a unit time-like vector field γ^i, the standard four-velocity V^i is defined as: $V^i \equiv dx^i/dT$, $dT = -\gamma_i\, dx^i/c$ being the relative standard time-interval [3], [4].

With respect to the metric (4), G^i_k is identically zero if $i \neq k$, which implies $S^i_k = 0$ for $i \neq k$ too. The remaining gravitational equations for $i = k$, conveniently combined, give rise to the following system of ordinary differential equations in the unknown functions $R(r)$, $u(r)$, $k(r)$:

$$
(5) \qquad \begin{cases} R'' = \chi(-g)^{\frac{1}{2}}(S^1_1 + S^3_3) \\ 2u'' + 2u' R'/R = \chi(-g)^{\frac{1}{2}}(S^1_1 + S^2_2 + S^3_3 - S^4_4)/R \\ u'^2 - k' R'/R + R''/R = \chi(-g)^{\frac{1}{2}} S^3_3/R \\ k'' + u'^2 = \chi(-g)^{\frac{1}{2}} S^2_2/R \end{cases}
$$

the prime denoting differentiation with respect to r. Together with (5) we have to consider the Maxwell vacuum field equations:

$$
(5') \qquad F^{li}_{;l} = 0 \quad , \quad F_{ik,l} + F_{kl,i} + F_{li,k} = 0 ,
$$

which on account of G_3-symmetry, become:

$$
(6) \qquad \tfrac{1}{2}(\eta_{iklm} F^{lm} \xi^i_{(u)} \xi^k_{(v)})_{,h} = 0 \quad , \quad (F_{ik} \xi^i_{(u)} \xi^k_{(v)})_{,h} = 0 ,
$$

respectively. In (6), the indices u, v take the values 2, 3, 4; $\eta_{iklm} = (-g)^{\frac{1}{2}} \varepsilon_{iklm}$ is the Levi-Civita's tensor, and for each region A_α one has to adopt the respective Killing vectors.

(5), (6) are the Einstein-Maxwell equations in the unknown functions $R(r)$, $u(r)$, $k(r)$, $F_{ij}(r)$.

In order to obtain the physical interpretation of the tensor components F_{ik}, we shall use its natural decomposition with respect to the frames S_α. That is, we shall call relative electric field and relative magnetic field with respect to S_α, the spatial vectors $E_i \equiv \gamma_{ir}\gamma_s F^{rs}$, $H^i \equiv \frac{1}{2}\eta^{ikh}\gamma_{kr}\gamma_{hs}F^{rs}$, respectively. Where γ is the unit time-like vector field defining S_α, $\gamma_{ir} \equiv (g_{ir} + \gamma_i\gamma_r)$ is the space-projection tensor, and $\eta^{ikh} \equiv \eta^{irkh}\gamma_r$ is the spatial Levi-Civita's tensor [1], [4].

Taking into account the conditions $S^i_k = 0$ for $i \neq k$, the equations (6) can be immediately solved: for each region A_α, there are three qualitatively different possibilities. That is, one can experience either a radial electric field $E^1 = -\Theta/(\det\gamma_{ij})^{\frac{1}{2}}$, or an azimuthal magnetic field $H_2 = -\Lambda/\gamma_4$, or an axial magnetic field $H_3 = -\Psi/\gamma_4$. Θ, Λ, Ψ, being constants. Observing that H_2 is necessarily associated with an axial current, absent in the present situation, we can put, without any loss of generality, $\Lambda = 0$. On account of the charge distribution hypothesized, there is no other alternative than to assume the presence, *in* A_1, *of an axial magnetic field, and in* A_2 *and* A_3, *of a radial electric field.*

In other words the solutions of the Maxwell equations (6) in the regions A_α are:

$$F^{1'2'} = \Psi/(-g)^{\frac{1}{2}} \cdots \text{ in } A_1 \quad , \quad F^{14} = \Theta/(-g)^{\frac{1}{2}} \cdots \text{ in } A_2 , \tag{7}$$
$$F^{\hat{1}\hat{4}} = \Delta/(-g)^{\frac{1}{2}} \cdots \text{ in } A_3$$

the remaining $F^{i'k'}$, F^{ik}, $F^{\hat{i}\hat{k}}$, being zero. Ψ, Θ, Δ, are (pseudo) scalar-valued constants to be determined by imposing the e.m. junction conditions on $\bar{\Sigma}$ and Σ. For the solutions (7) one has $S^{1'}_{1'} + S^{3'}_{3'} = 0$, $S^1_1 + S^3_3 = 0$, $S^{\hat{1}}_{\hat{1}} + S^{\hat{3}}_{\hat{3}} = 0$, so it is possible, in each region A_α, to assume $R(r) = r$ (cfr. (5)), and (5) reduce to three (only two independent), equations

$$\begin{cases} 2u'' + 2u'/r = \chi(-g)^{\frac{1}{2}}(S^2_2 - S^4_4)/r \\ u'^2 - k'/r = \chi(-g)^{\frac{1}{2}} S^3_3/r \cdots \text{ in } A_1, A_2, A_3, \\ k'' + u'^2 = \chi(-g)^{\frac{1}{2}} S^2_2/r \end{cases} \tag{8}$$

in two unknown functions: $u(r)$, $k(r)$. Of course for each region A_α one has to adopt the respective adapted coordinates and the respective values of S^i_k. In each region A_α (8), can easily be solved, each solution depending on three new constants: two to be determined by means of regularity and junction conditions, the third one being physically unessential. Disposing conventiently of these latter, the solution of (8), in each A_α, can be cast in the following form [2]:

A_1, *the inner region*:

$$ds^2 = \frac{(1 + hr'^2)^2}{(1 + ha^2)^2}(dr'^2 + dz'^2 - c^2\, dt'^2) + r'^2 \frac{(1 + ha^2)^2}{(1 + hr'^2)^2} d\varphi'^2 \tag{9}$$

where $h \equiv \chi\Psi^2/8$ and a is a constant homogeneous to a length. Notice that in (9) only a new essential constant a appear, the other one, present in the general solution, has been put to zero in order to realize full regularity in A_1.

A_2; *the intermediate region*:

$$ds^2 = \left(\frac{r}{p}\right)^{2b} \frac{[1 - H(r/p)^{-2b}]^2}{(1 - H)^2} \left[\left(\frac{r}{p}\right)^{2b^2}(dr^2 + dz^2) + r^2\, d\varphi^2\right] - \tag{10}$$
$$- c^2 \left(\frac{r}{p}\right)^{-2b} \frac{(1 - H)^2}{[1 - H(r/p)^{-2b}]^2}\, dt^2$$

where $H \equiv \chi\Theta^2/8\, b^2$, and p, b are new essential constants; p is homogeneous to a length, b—(*Levi-Civita's mass*)—is now different from zero, since it does not disturb the regularity in A_2.

A_3, *the outer region*:

$$\mathrm{d}s^2 = A^2 \left(\frac{\hat{r}}{q}\right)^{2B} \frac{[1 - K(\hat{r}/q)^{-2B}]^2}{(1-K)^2} \times \tag{11}$$

$$\times \left[\left(\frac{q}{p}\right)^{2b^2}\left(\frac{\hat{r}}{q}\right)^{2B^2}(\mathrm{d}\hat{r}^2 + \mathrm{d}\hat{z}^2) + \hat{r}^2\,\mathrm{d}\hat{\varphi}^2\right] -$$

$$- c^2 A^{-2}\left(\frac{\hat{r}}{q}\right)^{-2B} \frac{(1-K)}{[1 - K(\hat{r}/q)^{-2B}]^2}\,\mathrm{d}\hat{t}^2$$

where $A \equiv (q/p)^b\,[1 - H\,(q/p)^{-2b}]/(1-H)$ and $K \equiv \chi\Delta^2/8\,B^2$.

q and B are other essential constants: q is homogeneous to a length, B, is another *Levi-Civita's mass*, which, as b in A_2, has to be assumed different from zero.

3. Junction conditions for the gravitational potentials and the E. M. Fields

The gravitational junction conditions request, first of all, the continuity of the 3-dimensional metrics induced on $\bar{\Sigma}$ and Σ by the metrics of the contiguous regions A_1, A_2, and A_2, A_3, respectively [2]. That is,

$$g_{u'v'}(r_0)\,\mathrm{d}x^{u'}\,\mathrm{d}x^{v'} = g_{uv}(r_0)\,\mathrm{d}x^u\,\mathrm{d}x^v \cdots \text{ on } \bar{\Sigma}, \tag{12}$$

$$g_{uv}(R_0)\,\mathrm{d}x^u\,\mathrm{d}x^v = g_{\hat{u}\hat{v}}(R_0)\,\mathrm{d}x^{\hat{u}}\,\mathrm{d}x^{\hat{v}} \cdots \text{ on } \Sigma \qquad (u, v = 2, 3, 4). \tag{12'}$$

(12) yields $a = p = r_0$, and imposes three relations among the constants $\eta, \lambda, \sigma, \nu$, determining the connections (cfr. (3)), between the adapted coordinates $(x^{i'})$ and (x^i). According to such relations, one remarkably finds that (3) can be written in the "*Lorentz-like*" form:

$$r_0\,\varphi = \frac{r_0\,\varphi' + v(1,2)\,t'}{\sqrt{1 - v^2(1,2)/c^2}} \quad , \quad z = z' \quad , \quad t = \frac{\varphi'\,r_0\,v(1,2)/c^2 + t'}{\sqrt{1 - v^2(1,2)/c^2}}, \tag{13}$$

where $v(1,2) = -v(2,1) = r_0\,\omega(1,2)$ is the magnitude with sign, on $\bar{\Sigma}$, of the standard linear velocity of S_1 with respect to S_2. The scalar-valued constant $\omega(1,2)$, angular velocity of S_1 with respect to S_2, is given by $\omega(1,2) = -\omega(2,1) = c\xi_{(2)}\cdot\zeta_{(4)}(r_0)/\xi_{(2)}\cdot\zeta_{(2)}(r_0) = c\lambda/(1+\lambda^2 r_0^2)^{\frac{1}{2}}$. In a similar way (12') gives $q = R_0$ and allows (3') to be written as:

$$R_0 A^2\,\hat{\varphi} = \frac{R_0 A^2\varphi + w(2,3)\,t}{\sqrt{1 - w^2(2,3)/c^2}} \quad , \quad \hat{z} = z \quad , \quad \hat{t} = \frac{\varphi R_0 A^2 w(2,3)/c^2 + t}{\sqrt{1 - w^2(2,3)/c^2}}, \tag{14}$$

where $w(2,3) = -w(3,2) = R_0 A^2\,\omega(2,3)$ is the magnitude with sign, on Σ, of the standard velocity of S_2 with respect to S_3. The scalar-valued constant $\omega(2,3)$, angular velocity of S_2 with respect to S_3, is given by $\omega(2,3) = -\omega(3,2) = c\vartheta_{(2)}\cdot\xi_{(4)}(R_0)/\vartheta_{(2)}\cdot\xi_{(2)}(R_0) = c\beta/(1+\beta^2 R_0^2 A^4)^{\frac{1}{2}}$. The parameters λ and β, or preferably $v(1,2) = r_0 c\lambda/(1+\lambda^2 r_0^2)^{\frac{1}{2}}$ and $w(2,3) = R_0 A^2 c\beta/(1+\beta^2 R_0^2 A^4)^{\frac{1}{2}}$, still unknown, will be determined later when we deal with gravitational junction conditions of higher order.

According to (13) and (14), the coordinates systems $(x^{i'})$, (x^i), $(x^{\hat{i}})$, are physically admissible over regions larger than A_1, A_2, A_3, where, originally, they were introduced respectively. Beyond such larger regions one cannot define the stationary frames S_1, S_2, S_3. According to our statement of the E-R problem we have to characterize the dynamical state and the e.m. properties of the inner shell with respect to the frame S_3. This is possible only if S_3 extends over the region A_2 too. This implies the following limitations for the ratio of the radiuses of the shells: $|w(2,3)/c| < \left(\frac{r_0}{R_0}\right) A^{-2} < |c/w(2,3)|$.

If this condition held then one could obtain, on $\bar{\Sigma}$, the angular velocity of S_3 with respect to S_1:

$$\omega(3,1) = \frac{\omega(3,2)+\omega(2,1)}{1+r_0^2\,\omega(3,2)\,\omega(2,1)/c^2}. \tag{15}$$

Since the inner shell is at rest with respect to S_3, $v(3,1) = r_0\,\omega(3,1)$ can be interpreted as the magnitude with sign, on $\bar{\Sigma}$, of the standard linear velocity of the inner shell with respect to S_1.

The e.m. junction conditions that we have to impose on the hypersurfaces $\bar{\Sigma}$ and Σ can be obtained from Maxwell equations with sources, and can be written:

$$[F^{ki}]_{r_0}\,\bar{n}_i = \bar{J}^k(r_0) \quad , \quad [\eta^{kilm} F_{lm}]_{r_0}\,\bar{n}_i = 0 \cdots \text{on } \bar{\Sigma}\,, \tag{16}$$

$$[F^{ki}]_{R_0}\,n_i = J^k(R_0) \quad , \quad [\eta^{kilm} F_{lm}]_{R_0}\,n_i = 0 \cdots \text{on } \Sigma\,, \tag{16'}$$

where $\bar{n}_i = (1,0,0,0)$ and $n_i = (1,0,0,0)$ are the normal vectors to $\bar{\Sigma}$ and Σ respectively. The symbol [M] means the discontinuity of the quantity M on the hypersurface specified, e.g. $[M]_{r_0} = \lim_{\varepsilon\to 0^+}\{M(r_0+\varepsilon) - M(r_0-\varepsilon)\}\cdot$ $\bar{s}^k = \bar{J}^k\,\delta(\bar{\Sigma})$ and $s^k = J^k\,\delta(\Sigma)$ are the four-current densities evolving on $\bar{\Sigma}$ and Σ respectively, $\delta(\bar{\Sigma})$ and $\delta(\Sigma)$ just being invariant Dirac measures based on such hypersurfaces.

According to our hypotheses the inner shell is uniformly charged and at rest with respect to the frame S_3, that is

$$\bar{J}^k = \rho_0\,\underset{(4)}{\vartheta^k}/(-\underset{(4)}{\vartheta^h}\,\underset{(4)}{\vartheta_h})^{\frac{1}{2}} = (0\,,\,\rho_0\,\bar{w}(3,2)/r_0 c\,,\,0\,,\,\rho_0)/(1-\bar{w}(3,2)^2/c^2)^{\frac{1}{2}}, \tag{17}$$

ρ_0 being the proper surface charge density of the inner shell. $\bar{w}(3,2) = r_0\,\omega(3,2)$ is the magnitude with sign, on $\bar{\Sigma}$, of the standard linear velocity of S_3, (*hence of the inner shell*), with respect to S_2.

The outer shell is uncharged and in uniform rotation with respect to S_3. This implies only $J^k \underset{(4)}{\vartheta_k} = 0$, which establishes one relation among the four components of J^k, hence, three among them are, a priori, available. Introducing (17) in the right member of (16) and taking into account the previous constraint, one obtains:

$$(18) \qquad \Psi = \rho' \left[v(3,1) - v(2,1)\right]/c \quad , \quad \Theta = -\rho r_0 \left[1 - v(1,2)\,\bar{w}(3,2)/c^2\right],$$

$$(18') \qquad \Delta = \Theta\left(1 - w^2(2,3)/c^2\right)^{-\frac{1}{2}},$$

$$(19) \qquad J^{\hat{2}} = -\frac{\bar{w}(3,2)}{c}\left(\frac{r_0}{R_0}\right)^{1+2b^2} A^{-2}\,\Delta .$$

ρ' and ρ are the relative surface charge densities of the inner shell with respect to S_1 and S_2 respectively:

$$\rho' = -\underset{(4)}{\zeta^k}\,\bar{J}_k/(-\underset{(4)}{\zeta^h}\,\underset{(4)}{\zeta_h})^{\frac{1}{2}} = \rho_0\left[1 - v(3,1)^2/c^2\right]^{-\frac{1}{2}}$$

$$\rho = -\underset{(4)}{\xi^k}\,\bar{J}_k/(-\underset{(4)}{\xi^h}\,\underset{(4)}{\xi_h})^{\frac{1}{2}} = \rho_0\left[1 - \bar{w}(3,2)^2/c^2\right]^{-\frac{1}{2}}.$$

At first sight the expressions (18) and (18′) bear evidence, in a particular clear way, of the connection among the pseudo-scalars Ψ, Θ, Δ, describing the e.m. fields in A_1, A_2, A_3, and the scalar $\omega(\alpha, \mu)$ associated to the mutual rotation of the frames S_α. The constants $\omega(\alpha, \mu)$, together with b and B, still unknown, will be determined later by means of the gravitational junction conditions of higher order. From (19) comes out, according to our assumptions, that, with respect to S_3, a current flows in the outer shell. Such a current is absent only if $\omega(3,2) = 0$. That is, as we shall see later, if the outer shell is at rest or if we do not take into account the gravitational interactions ($\chi = 0$). Hence, in order that our statement of the E-R problem makes sense, we have to suppose that the current (19) is initially flowing in the outer shell. Moreover this latter should be a perfect conductor, otherwise the current, suitably supplied, will tend to zero by heating, wia the Joule effect, the outer shell, and we can reasonably infer that its angular velocity would decrease till the reciprocal rest between the shells. The presence of such a current is the price we have to pay in order that the assumed staticity of the frames S_α holds in the regions A_α.

We can now formulate properly the structural hypotheses about the thin shells evolving on the hypersurfaces $\bar{\Sigma}$ and Σ, and to impose the gravitational junction conditions taking into account such energetic structures. We shall deal with such considerations and with the inferences that may be drawn from them, in the second part of this paper.

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