Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **64** (1978), n.1, p. 30–31.

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<http://www.bdim.eu/item?id=RLINA_1978_8_64_1_30_0>

Analisi matematica. — *Perturbations preserving asymptotics of spectrum.* Nota di ALEXANDER G. RAMM, presentata (*) dal Corrisp. G. FICHERA.

RIASSUNTO. — Vengono annunziati alcuni risultati relativi al problema consistente nel pertubare un dato operatore lineare in modo che gli autovalori dell'operatore perturbato siano asintotici a quelli dell'operatore dato.

1. Let H be a separable Hilbert space, A be a closed densely defined linear operator with discrete spectrum, A^{-1} be a bounded operator, T be a linear operator $D(T) \supset D(A)$, $D(A) = \text{Dom}\, A$, $R(A) = \text{im}(A)$, $N(A) = \text{Ker}\, A$, $\{0\}$ the set consisting of 0, $C = A^{-1} T$, $|f| = (f, f)^{\frac{1}{2}}$, $S_n(A)$, $\lambda_n(A)$ be respectively, the s-numbers [1; II, § 2, § 7] and the eigenvalues of A. If $A \geq m > 0$ let H_A be the Hilbert space which is the completion of D (A) with respect to the norm $\|f\| = (Af, f)^{\frac{1}{2}}$. We call the spectrum of a linear operator A discrete if every its point is an isolated eigenvalue of finite algebraic multiplicity [1]. Let $B = A + T$, $D(B) = D(A)$. We announce some results sufficient for the limit relation $\lambda_n(B)\lambda_n^{-1}(A) \to 1$, or $S_n(B) S_n^{-1}(A) \to 1$, $n \to \infty$ to be valid. The results generalize some known results [1; V, § 11]. The methods of the proofs differ from these of [1]. All the operators below are linear operators in the Hilbert space H.

2. THEOREM 1. *Let* Q, S *be some compact operators,* $\dim R(Q) = \infty$, $N(I + S) = \{0\}$. *Then* $S_n(Q + QS) S_n^{-1}(Q) \to 1$, $S_n(Q + SQ) S_n^{-1}(Q) \to 1$, $n \to \infty$.

THEOREM 2. *Let the operators* C, A^{-1}, TA^{-1} *be compact,* $N(B) = \{0\}$. *Then the spectrum of operator* B *is discrete,* $S_n(B) S_n^{-1}(A) \to 1$, $n \to \infty$. *If in addition* $A \geq m > 0$ *and the operator* B *is normal, then* $\lambda_n(B) \lambda_n^{-1}(A) \to 1$, $n \to \infty$.

THEOREM 3. *Let* $A \geq m > 0$, $D(T) \supset H_A$, C *is compact in* H_A *and* $B = B^*$. *Then the spectrum of* B *is discrete and* $\lambda_n(B) \lambda_n^{-1}(A) \to 1$, $n \to \infty$.

3. Let $A[f, f]$ be a positive definite quadratic form (q.f.) on H. A real valued quadratic form $T[f, f]$ defined on D [A] is called compact relatively to q.f. $A[f, f]$ if the set $\{f : A[f, f] \leq 1\}$ contains a sequence f_n such that $T[f_n - f_m, f_n - f_m] \to 0$, $n, m \to \infty$. The spectrum of a semibounded q.f. is the spectrum of the corresponding selfadjoint operator.

(*) Nella seduta del 14 gennaio 1978.

THEOREM 4. *Let* $A[f,f]$ *be a positive definite* q.f. *with discrete spectrum* $\lambda_n(A)$, *a real-valued* q.f. $T[f,f]$ *is compact relative to* $A[f,f]$. *Then the* q.f. $B[f,f] = A[f,f] + T[f,f]$, $D[B] = D[A]$ *has discrete spectrum*, $\lambda_n(B)\lambda_n^{-1}(A) \to 1$, $n \to \infty$.

Example 1. Let $H = L^2(D)$, $D \subset R^m$ is a bounded domain with the smooth boundary Γ, $A[f,f] = \int_D \{|\nabla f|^2 + |f|^2\} dx$, $T[f,f] = \int_\Gamma \sigma(s)|f(s)|^2 ds$, $\sigma(s) \subset C^1(\Gamma)$. The q.f. $T[f,f]$ is compact relatively to $A[f,f]$.

So $\lambda_n(B)\lambda_n^{-1}(A) \to 1$, $n \to \infty$. Here $\{\lambda_n(A)\}$ is the spectrum of the Neumann problem in the domain D, $\{\lambda_n(B)\}$ is the spectrum of the following problem: $-\Delta f + f = \mu f$ in D, $\partial f/\partial N + \sigma(s) f = 0$ on Γ.

Example 2. Consider the problem $\mathscr{L}u = \lambda u$ in D, $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1$, $\mathscr{L}_0$ being a selfadjoint elliptic differential operator of order $2r$, $\mathscr{L}_1$ being a differential operator of order $r_1 < 2r$ in $H = L^2(D)$.

Suppose $N(\mathscr{L}_0) = \{0\}$. The operator $\mathscr{L}_0^{-1}\mathscr{L}_1$ is compact in H. So according to Theorem 2 $S_n(\mathscr{L})S_n^{-1}(\mathscr{L}_0) \to 1$, $n \to \infty$. If in addition $\mathscr{L}$ is normal then $\lambda_n(\mathscr{L})\lambda_n^{-1}(\mathscr{L}_0) \to 1$, $n \to \infty$.

REFERENCE

[1] I. C. GOHBERG and M. G. KREIN (1965) – *Introduction to the theory of linear nonselfadjoint operators*, «Nauka, Moskwa», 1965 (in Russian). English translation, «American Math. Society», *18*, Translations of Mathematical Monographs, 1969.