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Linear stochastic differential equations in Hilbert spaces

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Analisi matematica. — *Linear stochastic differential equations in Hilbert spaces* (*). Nota di GIUSEPPE DA PRATO, MIMMO IANNELLI e LUCIANO TUBARO, presentata (**) dal Corrisp. G. STAMPACCHIA.

RIASSUNTO. — Si studiano risultati di esistenza e regolarità delle traiettorie per la soluzione di equazioni differenziali stacastiche lineari in uno spazio di Hilbert.

1. INTRODUCTION

Let $(\Omega, \mathcal{E}, P)$ be a probability space, $w_t, t \in [0, T]$ be a real Wiener process in it. Let $\mathcal{F}_t$ be a family of σ-algebras non anticipating with respect to w_t. Let H be a Hilbert space and $A : D_A \to H$; $B : D_B \to H$ two linear operators in H.

We want to study the following equation:

$$u(t) = u_0 + \int_0^t (Au(s) + f(s))\, ds + \int_0^t (Bu(s) + g(s))\, dw_s \tag{1}$$

where A is the infinitesimal generator of a c_0-semigroup.

By the use of the properties of the Itô integral and by the usual methods of the contraction principle, we prove the existence of a solution to (1) in the space $C(0, T ; L^2(\Omega, H))$, (see [2]).

On the other hand by regularity results (see [3]) for abstract differential equations together with the continuity properties of the Itô integral we also prove continuity and Hölder continuity of trajectories of the solutions.

The interest of this latter result is also concerned with the study of the existence of a maximal solution for semi-linear abstract stochastic differential equations, by methods similar to those used in [4].

This latter kind of equations will be studied in a forthcoming paper.

The results of this paper are stated in section 3, while section 2 is devoted to a preliminar study of the properties of the process:

$$X(t) = \int_0^t \exp((t-s)A)\, g(s)\, dw_s . \tag{2}$$

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2. THE PROPERTIES OF X

By first we define, as usual, the following space:

$M_2 = \Big\{$the set of all measurable processes u in $[0, T]$ with values in H, adapted to the family $\mathcal{F}$, and such that $\int_0^T E|u|^2 ds < +\infty \Big\}$ then we put:

$$Z(0, T; H) = C(0, T; L^2(\Omega, H)) \cap M_2 \tag{3}$$

$Z(0, T; H)$ is a closed subspace of $C(0, T; L^2(\Omega, H))$ with the usual norm:

$$\|u\| = \sup_{0 \le t \le T} (E|u|^2)^{\frac{1}{2}}$$

so that we endowe Z with this norm. Yet by $Z(0, T; D_A)$ we denote those processes $u \in Z(0, T; H)$ such that $u \in D_A$ and $Au \in Z(0, T; H)$. For $g \in Z(0, T; H)$ $(Z(0, T; D_A))$ the integral in (2) makes a sense for $t \in [0, T]$, so that the process X is defined. We want to investigate the properties of X under various assumptions on A and g. Firstly we have:

PROPOSITION 1. *Let* A *be the infinitesimal generator of a* c_0*-semigroup on* H, *then*:

$$g \in Z(0, T; H) \; (\textit{resp.}\ Z(0, T; D_A)) \Rightarrow X \in Z(0, T; H) \; (\textit{resp.}\ Z(0, T; D_A)). \tag{4}$$

Proof. Let $A_n = n^2 R(n, A) - n = AR(n, A)\, n$ be the Yosida approximation for A, put:

$$X_n(t) = \int_0^t \exp((t-s) A_n)\, g(s)\, dw_s \tag{5}$$

X_n is the unique solution of the stochastic equation:

$$X_n(t) = \int_0^t A_n X_n(s)\, ds + \int_0^t g(s)\, dw_s \tag{6}$$

as it can be easily verified, observing that

$$X_n(t) = \exp(tA_n) \int_0^t \exp(-sA_n)\, g(s)\, dw_s .$$

Moreover it is

$$\forall t \; X_n(t) \to X(t) \qquad \text{in probability,} \tag{7}$$

in fact $\forall t \in [0, T]$ and $\omega \in \Omega$:

$$\exp((t-s) A_n) g(s) \to \exp((t-s) A) g(s) \quad \text{in} \quad L^2(0, t; H)$$

(7) implies that X is a measurable process, adapted to the family $\mathcal{F}_t$. On the other hand

$$\int_0^T E|X(s)|^2 ds = \int_0^T E\left|\int_0^t \exp((t-s) A) g(s) dw_s\right|^2 dt =$$

$$= \int_0^T dt \int_0^t E|\exp((t-s) A) g(s)|^2 ds \le M^2 \int_0^T dt \int_0^t E|g(s)|^2 ds^{(1)} < +\infty$$

so that $X \in M_2$. Finally it is:

$$E|X(t) - X(t_0)|^2 = E\left|\int_{t_0}^t \exp((t-s) A) g(s) dw_s + \right.$$

$$\left. + \int_0^{t_0} (\exp((t-s) A) - \exp((t_0-s) A)) g(s) dw_s\right|^2 \le$$

$$\le 2 M^2 \int_{t_0}^t E|g(s)|^2 ds + 2 \int_0^{t_0} |(\exp((t-s) A) -$$

$$- (\exp((t_0-s) A)) g(s)|^2 ds \to 0$$

as $t \to t_0$ so that $X \in C(0, T; L^2(\Omega, H))$.

All this means that $X \in Z(0, T; H)$. If now $g \in Z(0, T; D_A)$ it is $AX(t) = \int_0^t \exp((t-s) A) Ag(s) dw_s$ and in the same way as before we can prove that $X \in Z(0, T; D_A)$.

The previous proposition does not give any information on the continuity of the process X. To get this kind of results we need stronger hypotheses either on A or on g.

First of all we remark the following equality:

$$X_n(t) = \gamma(t) + A_n \int_0^t \exp((t-s) A_n) \gamma(s) ds \tag{8}$$

where X_n is defined in (5) and $\gamma(t) = \int_0^t g(s) dw_s$.

(1) $M = \sup_{0 \le t \le T} |e^{tA}|$.

Indeed it can be easily proved that the right hand side of (8) is the unique solution of (6). Then we have:

PROPOSITION 2. *Let* A *be the infinitesimal generator of a* c_0*-group then*:

$$g \in Z(0, T; H) \Rightarrow X \quad \textit{is continuous in } H$$

$g \in Z(0, T; D_A) \Rightarrow X$ is *α-Hölder continuous in* H $(\alpha < \frac{1}{2})$ *and continuous in* D_A.

Proof. The proof easily follows from the properties of the Itô integral, as it is:

$$X(t) = e^{tA} \int_0^t e^{-sA} g(s)\, ds . \tag{9}$$

In the general case for A it is necessary to suppose g regular:

PROPOSITION 3. *Let* $g \in Z(0, T; D_A)$ *then* X *is an α-Hölder continuous process in* H $(\alpha < \frac{1}{2})$.

Proof. Owing to the hypotheses $\gamma(t) = \int_0^t g(s)\, ds$ is in D_A and $A\gamma(t) = \int_0^t Ag(s)\, ds$. Going to the limit in (8) (see (7)) we get:

$$X(t) = \gamma(t) + \int_0^t \exp((t-s)A)\, A\gamma(s)\, ds . \tag{10}$$

Now the processes γ and $A\gamma$ are Hölder continuous in H (2), moreover by a standard result (see [3]) the second term on the right hand side of (10) is also a Hölder continuous process. Thus (6) is proved.

Finally we have:

PROPOSITION 4. *Let* A *be the infinitesimal generator of an analytic semigroup on* H; *then.*

$$g \in Z(0, T; H) \Rightarrow X \textit{ is an α-Hölder continuous } (\alpha < \tfrac{1}{2}) \textit{ process in } H.$$

Proof. In this case as the process γ is α-Hölder continuous $(\alpha < \frac{1}{2})$ in H, by a result in [3] it follows that the process

$$t \to \int_0^t \exp((t-s)A)\, \gamma(s)\, ds$$

(2) This is a well-known fact, see for istance [5].

is α-Hölder continuous ($\alpha < \frac{1}{2}$) in D_A and moreover it is

$$A_n \int_0^t \exp((t-s)A_n)\,\gamma(s)\,ds \to A \int_0^t \exp((t-s)A)\,\gamma(s)\,ds\,,$$

so that going to the limit in (8) we have

$$X(t) = \gamma(t) + A \int_0^t \exp((t-s)A)\,\gamma(s)\,ds$$

and X is α-Hölder continuous ($\alpha < \frac{1}{2}$) in H.

Remark. Under the assumptions of Proposition 4 actually it can be proved that $X \in Z(0, T; D_A)$; the proof of this fact is perfectly similar to that of Proposition 1.

3. Linear stochastic differential equations

We now study the stochastic differential equation (1) in the following generalized form:

$$(11) \qquad u(t) = \exp(tA)\,u_0 + \int_0^t \exp((t-s)A)\,f(s)\,ds + \int_0^t \exp((t-s)A)\,(Bu(s) + g(s))\,dw_s\,.$$

(11) is equivalent to (1) when $A \in \mathcal{L}(H)$.

PROPOSITION 4. *Let* A *be the infinitesimal generator of a c_0-semi-group on* H, $B \in \mathcal{L}(H)$, $f, g \in Z(0, T; H)$, $u_0 \in L^2(\Omega, H)$, *$\mathcal{F}_0$-measurable. Then* (17) *has a unique solution* $u \in Z(0, T; H)$. *Moreover it is*:

$$(12) \qquad f, g \in Z(0, T; D_A),\ B(D_A) \subset D_A,\ u_0 \in L^2(\Omega, D_A) \Rightarrow u \text{ is a solution of } (1).$$

(13) A *is the infinitesimal generator of a c_0-group on* H

$\Rightarrow$ *u is a continuous process in* H.

(14) A *is the infinitesimal generators of an analytic semigroup in* H

$u_0 \in L^2(\Omega, D_{A^\theta})$, $\theta \in (0, \frac{1}{2})$

$\Rightarrow$ *u is α-Hölder continuous process in* H [(3)].

(3) If $u_0 \in L^1(\Omega, H)$ then u is Hölder continuous only for $t \in (0, T]$.

Proof. Put:

$$(15) \qquad v(t) = \exp(tA)\, u_0 + \int_0^t \exp((t-s)A) f(s)\, ds + \int_0^t \exp((t-s)A)\, g(s)\, dw_s$$

$$(16) \qquad u \to \phi(u) = \int_0^t \exp((t-s)A)\, Bu(s)\, dw_s .$$

By the Proposition 1 $v \in Z(0, T; H)$ and $\phi : Z(0, T; H) \to Z(0, T; H)$, then (11) can be written as the following equation in $Z(0, T; H)$:

$$(17) \qquad u = \phi(u) + v .$$

Now ϕ is a linear mapping such that

$$\|\phi(u)\|^2 \leq TM^2 |B|^2 \|u\|^2$$

so that by the standard argument of contraction principle, by the existence of a unique solution in $Z(0, T; H)$ follows. To prove (12) we first remark that with the assumptions in (12) the same argument used before can be used to show the existence of a solution $u \in Z(0, T; D_A)$, then putting:

$$v_n = \exp(tA_n)\, u_0 + \int_0^t \exp((t-s)A_n) f(s)\, ds + \int_0^t \exp((t-s)A_n)(Bu(s) + g(s))\, dw_s$$

it is $\forall t \in [0, T]$

$$v_n \to u \quad \text{in } L^2(\Omega, H)$$

$$Av_n \to Au \quad \text{in } L^2(\Omega; H)$$

and

$$v_n = u_0 + \int_0^t (A_n v_n(s) + f(s))\, ds + \int_0^t (Bu(s) + g(s))\, dw_s$$

so that (1) follows going to the limit. Finally (13) and (14) follow directly from Propositions 2 and 3.

The previous existence result can be extended to the case of B unbounded, assuming that $D_{A^\theta} \subset D_B$. To do this it is necessary to state the following proposition whose proof is quite similar to that of Proposition 1 [4].

(4) It is sufficient to recall the following estimate

$$|B \exp(tA)| \leq K |BA^{\theta-1}| t^{\theta-1} \qquad t > 0 .$$

PROPOSITION 5. *Let* A *be the infinitesimal generator of an analytic semigroup and* $B : D_B \to H$ *be such that* $D_{A^\theta} \subset D_B$ $(\theta c] 0, \frac{1}{2} [)$, *then if* $g \in Z(0, T; H)$ *then the process*

$$Y(t) = \int_0^t B \exp((t-s)A) g(s) ds$$

is in $Z(0, T; H)$.

Then we have:

PROPOSITION 6. *Let* A *be the infinitesimal generator of an analytic semigroup and* B *an invertible operator such that* $D_{A^\theta} \subset D_B$ *for some* $\theta \in] 0, \frac{1}{2} [$. *If* $f, g \in Z(0, T; H)$ *and* $u_0 \in L^2(\Omega, D_{A^\theta})$ *is* $\mathcal{F}_0$*-measurable then* (11) *has a unique solution* $u \in Z(0, T; H)$. *Moreover* u *is an* α*-Hölder continuous process* $(\alpha < \frac{1}{2})$ *in* H.

Proof. Consider the following equation in $Z(0, T; H)$:

$$(18) \qquad \left\{ \begin{aligned} v(t) = B \exp(tA) u_0 + \int_0^t B \exp((t-s)A) f(s) ds + \\ + \int_0^t B \exp((t-s)A) v(s) dw_s . \end{aligned} \right.$$

By Proposition 5, proceeding as in Proposition 4 it can be showed the existence of a unique solution $v \in Z(0, T; H)$ of (18).

Then $u = B^{-1} v \in Z(0, T; D_B)$ is the solution of (11) and by Proposition 4 is an α-Hölder continuous process in H.

Remark 2. The assumptions on B in Proposition 5 seem to be rather strong, actually if the domains of A and B are not comparable there is not existence in general. A typical case is equation (1) with $A = 0$. Indeed if B is hermitian and bounded it is easy to prove that:

$$(19) \qquad u(t) = \exp(Bw(t) - B^2 t/2)$$

if B is unbounded then any solution must be of the form (19), but in order to (19) be meaningful for every u_0 it is necessary that either B be the infinitesimal generator of a c_0-group, either $-B^2$ be a generator of a c_0 semigroup.

This is, in general, impossible by a simple argument on the spectrum of B.

Remark 3. Propositions 4 and 6 have been proved assuming $u_0 \in L^2(\Omega, H)$ or $u_0 \in L^2(\Omega, D_{A^\theta})$ and $f, g \in Z(0, T; H)$.

The results are still true when u_0 is only $\mathcal{F}_0$-measurable (with values in H or D_{A^θ} respectively) and f, g continuous processes in M_2. The proof of this can be carried through as in [4], Theorem 4.

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