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Correspondence between the class of left nonassociative C-rings and a class of loops

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 14 gennaio 1978

Presiede il Presidente della Classe ANTONIO CARRELLI

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Algebra. — *Correspondence between the class of left nonassociative C-rings and a class of loops.* Nota di MIRELA STEFĂNESCU, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Estendendo risultati precedenti di Malcev, di Weston e dell'autrice, si dimostra che esiste una corrispondenza tra la classe dei C-anelli non associativi sinistri e una classe di cappi. Tale corrispondenza è anche un'equivalenza tra le teorie formalizzate di dette classi.

There is a correspondence between the class of nonassociative rings and a class of nilpotent groups, which is also an equivalence between their formalized theories. K. Weston [10] constructed it, generalizing an idea of Mal'cev [5] for the class of nonassociative rings with identity. We obtained a more general result, for a special class of distributive nonassociative near-rings (with $x \cdot y + z = z + x \cdot y$, for all x, y) and a larger class of groups. This is the largest class of nonassociative near-rings which corresponds to a class of groups. We gave this result in [8], and proved there that the established correspondence is an equivalence between their formalized theories and between the categories which have the above classes, as classes of objects, and the near-ring homomorphisms and, respectively, group homomorphisms, as morphisms.

The purpose of this paper is to construct a similar correspondence between the class of left nonassociative C-rings and a class of loops. We show also

(*) Nella seduta del 14 gennaio 1978.

that this is an equivalence between the formalized theories of these classes. The correspondences from [8], hence those from [5] and [10], as well as some other correspondences, are obtained from the one given here, as its restrictions.

1. Definitions and notations

A *left nonassociative near-ring* is a triple $(N, +, \cdot)$, such that $(N, +)$ is a group, and $\cdot$ is left distributive over $+$. If $0 \cdot x = 0$ for all $x \in N$, then N is called a *left nonassociative* C-*ring* [1, § 4 (b)]. If, in addition, $(-x) \cdot y = -x \cdot y$, for all $x, y \in N$, then we call N a *strict* (*left nonassociative*) C-*ring*. N is called a *distributive near-ring*, if $\cdot$ is also right distributive over $+$. Obviously, a distributive near-ring is a strict C-ring, and, thus, a C-ring.

We use the following notations: $\mathscr{C}$-the class of all left nonassociative C-rings; $\mathscr{C}_1$-its subclass made up of strict C-rings; $\mathscr{D}$-the subclass of $\mathscr{C}_1$ made up of distributive near-rings; $\mathscr{D}_1$-the subclass of $\mathscr{D}$ of distributive near-rings N in which $x \cdot y + z = z + x \cdot y$, for all $x, y \in N$.

Note that $\mathscr{C}$, as the class of objects, together with the near-ring homorphisms, as morphisms, forms a category, $\tilde{\mathscr{C}}$, with $\tilde{\mathscr{C}}_1$, $\tilde{\mathscr{D}}$ and $\tilde{\mathscr{D}}_1$, as full subcategories.

An approach to the theory of near-rings can be found in [4]. For the definitions and notations concerning loops, see Bruck [2]. We use here the additive notation for the loop operation.

If $(L, +, 0)$ is a loop, then the sets

$$K_\lambda = \{a \mid a \in L, (a + x) + y = a + (x + y), \forall x, y \in L\},$$

$$K_\mu = \{a \mid a \in L, (x + a) + y = x + (a + y), \forall x, y \in L\},$$

$$K_\rho = \{a \mid a \in L, (x + y) + a = x + (y + a), \forall x, y \in L\}$$

are nonempty sets (because of the existence of 0) and they are called, respectively, the *left nucleus*, the *middle nucleus* and the *right nucleus* of L (see [2, p. 57]). All of them are subgroups of L.

It is known that for an additive operator on a loop L, $\alpha : L \to L$, (an endomorphism of L), $\alpha(0) = 0$ and $\operatorname{Ker} \alpha = \{x \mid x \in L, \alpha(x) = 0\}$ is a normal subloop of L [2, p. 60].

Denote by $\mathscr{L}$ *the class of loops satisfying the axioms* (*i*)-(*v*):

(i) *There exist two endomorphisms of* L, α *and* β, *such that* $\alpha \circ \alpha = \beta \circ \beta = \alpha \circ \beta = \beta \circ \alpha = 0$ (the null endomorphism of L).

(ii) *Denote* $A = \operatorname{Ker} \alpha = \{x \mid x \in L, \alpha(x) = 0\}$, $B = \operatorname{Ker} \beta = \{x \mid x \in L, \beta(x) = 0\}$ *and* $H = A \cap B$. *Then* $B \subseteq K_\rho$.

Remark 1.1. A is a subloop of L, while B and H are subgroups of L. Indeed, for any $a, b \in A$, the equations $a + x = b$ and $y + a = b$ have unique solutions in A, since $\alpha(a + x) = \alpha(b)$, $\alpha(y + a) = \alpha(b)$, $\alpha(a) = \alpha(b) = 0$ imply $\alpha(x) = \alpha(y) = 0$. We use the same argument for B and H. Now, the inclusions $H \subseteq B \subseteq K_\rho$ and the fact that K_ρ is a subgroup imply that B and H are subgroups.

(iii) *There exist two homorphisms* $\tilde{\alpha} : H \to B$, $\tilde{\beta} : H \to A$, *such that* $(\alpha \circ \tilde{\alpha})(x) = (\beta \circ \tilde{\beta})(x) = x$, *for all* $x \in H$.

Remark 1.2. Obviously, $(\alpha \circ \tilde{\beta})(x) = (\beta \circ \tilde{\alpha})(x) = 0$, for all $x \in H$. From the definitions of $\tilde{\alpha}$ and H, it follows that $\tilde{\alpha}(H) \subseteq K_\rho$ and $H \subseteq K_\rho$.

(iv) $\tilde{\beta}(H) \subseteq K_\lambda \cap K_\mu$, $H \subseteq K_\lambda$.

(v) H *and* $\tilde{\alpha}(H)$, *as well as* H *and* $\tilde{\beta}(H)$, *permute elementwise.*

Denote by x' the inverse of x, for any $x \in H$, hence $x + x' = x' + x = 0$. Denote by $[x, y]$ the unique solution of the equation:

$$x + y = (y + x) + [x, y], \qquad \forall x, y \in L. \tag{1.1}$$

LEMMA 1.1. *Let* $L \in \mathcal{L}$ *and* $H \subseteq L$. *For any* $x, y \in H$, *the elements* $[\tilde{\alpha}(x), \tilde{\beta}(y)]$ *and* $[\tilde{\beta}(y), \tilde{\alpha}(x)]$ *are in* H.

Proof. Denote $[\tilde{\alpha}(x), \tilde{\beta}(y)]$ by c. We have, indeed, $\tilde{\alpha}(x) + \tilde{\beta}(y) = (\tilde{\beta}(y) + \tilde{\alpha}(x)) + c$, and, by applying α and β, we obtain: $\alpha(c) = \beta(c) = 0$, hence $c \in H$. With a similar argument, we prove the second statement of the Lemma 1.1.

Now, applying properties of $\tilde{\alpha}(x)$ and $\tilde{\beta}(y)$, for all $x, y \in H$, given by axioms (iii)–(v) and Remarks 1.1 and 1.2, we obtain two forms for $[\tilde{\alpha}(x), \tilde{\beta}(y)]$, namely:

$$[\tilde{\alpha}(x), \tilde{\beta}(y)] = (\tilde{\alpha}(x') + \tilde{\beta}(y')) + (\tilde{\alpha}(x) + \tilde{\beta}(y)) \tag{1.2}$$

$$[\tilde{\alpha}(x), \tilde{\beta}(y)] = (\tilde{\beta}(y') + \tilde{\alpha}(x)) + (\tilde{\beta}(y) + \tilde{\alpha}(x')). \tag{1.3}$$

Indeed, from the equation $\tilde{\alpha}(x) + \tilde{\beta}(y) = (\tilde{\beta}(y) + \tilde{\alpha}(x)) + c$, by adding $\tilde{\beta}(y')$, which belongs to K_λ, to the left-hand side, we obtain: $\tilde{\beta}(y') + (\tilde{\alpha}(x) + \tilde{\beta}(y)) = \tilde{\alpha}(x) + c$ (since $\tilde{\beta}$ is an additive operator). Now, by adding $\tilde{\alpha}(x')$ to the left-hand side of the obtained equation, we have (1.2), since $\tilde{\alpha}$ is an additive operator, B is a group and $\tilde{\beta}(y') \in K_\mu$. From the same equation: $\tilde{\alpha}(x) + \tilde{\beta}(y) = (\tilde{\beta}(y) + \tilde{\alpha}(x)) + c$, by adding $\tilde{\beta}(y')$ to the left-hand side, and $\tilde{\alpha}(x')$ to the right-hand side, we have (1.3), since $\tilde{\beta}(y') \in K_\lambda$ and $\tilde{\alpha}(x') \in K_\rho$, while $[\tilde{\alpha}(x), \tilde{\beta}(y)]$ and $\tilde{\alpha}(x')$ permute.

Denote by $\mathcal{L}_1$ the subclass of $\mathcal{L}$ containing the loops L which satisfy the axiom:

(vi) *For any* $x \in H$ *and* $y \in L$, $\tilde{\alpha}(x) + (\tilde{\alpha}(x') + y) = y$. (We say that L satisfies the inverse property with respect to $\tilde{\alpha}(H)$).

Denote by $\mathcal{L}_2$ the subclass of $\mathcal{L}_1$ containing those loops L which satisfy the axiom:

(vii) $\tilde{\alpha}(H) \subseteq K_\mu$.

Denote by $\mathscr{G}$ the subclass of $\mathscr{L}_2$ containing those loops L which satisfy the axiom:

(viii) L *is a group.*

Remark 1.3. In this last case, some of the axioms are superfluous, as one can easily see.

Remark 1.4. $\mathscr{L}$, as the class of objects (that is, the objects are $(L, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$), together with the loop homomorphisms $\varphi : L \to L'$ ($\forall L, L' \in \mathscr{L}$), such that the following four diagrams:

$$(1.4) \qquad \begin{array}{ccc} L & \xrightarrow{\varphi} & L' \\ \alpha\downarrow & & \downarrow\alpha' \\ L & \xrightarrow{\varphi} & L' \end{array} \;,\quad \begin{array}{ccc} L & \xrightarrow{\varphi} & L' \\ \beta\downarrow & & \downarrow\beta' \\ L & \xrightarrow{\varphi} & L' \end{array} \;,\quad \begin{array}{ccc} H & \xrightarrow{\varphi|_H} & H' \\ \tilde{\alpha}\downarrow & & \downarrow\tilde{\alpha}' \\ L & \xrightarrow{\varphi} & L' \end{array} \;,\quad \begin{array}{ccc} H & \xrightarrow{\varphi|_H} & H' \\ \tilde{\beta}\downarrow & & \downarrow\tilde{\beta}' \\ L & \xrightarrow{\varphi} & L' \end{array}$$

are commutative, forms a category $\tilde{\mathscr{L}}$, with $\tilde{\mathscr{L}}_1, \tilde{\mathscr{L}}_2$ and $\tilde{\mathscr{G}}$ as full subcategories. (It is clear that $\varphi(x) \in H'$, for every $x \in H$, hence $\varphi|_H$ is a group homomorphism from H to H'. Remark 1.4 can be immediately verified).

LEMMA 1.2. *If* $L \in \mathscr{L}_1$, *then*:

$$[\tilde{\alpha}(x'), \tilde{\beta}(y)] = ([\tilde{\alpha}(x), \tilde{\beta}(y)])' = [\tilde{\alpha}(x), \tilde{\beta}(y')] .$$

Proof. Keep the notation $c = [\tilde{\alpha}(x), \tilde{\beta}(y)]$. To prove the first equality, we add, in turn, c' to the right-hand side, $\tilde{\alpha}(x')$ and $\tilde{\beta}(y')$ to the left-hand side of the equation $\tilde{\alpha}(x) + \tilde{\beta}(y) = (\tilde{\beta}(y) + \tilde{\alpha}(x)) + c, \forall x, y \in H$. Because of the properties: $c' \in K_\rho$, (vi), $\tilde{\beta}(y') \in K_\lambda$, (1.3), we obtain the desired equality. The second equality holds for any $L \in \mathscr{L}$. Indeed, by adding $(\tilde{\beta}(y') + \tilde{\alpha}(x'))$ to the right-hand side of the equation: $(\tilde{\alpha}(x) + \tilde{\beta}(y)) + c' = \tilde{\beta}(y) + \tilde{\alpha}(x), \forall x, y \in H$, we obtain the last equality. This is because of the properties $\tilde{\beta}(y') \in K_\mu$, (v), (iv), the fact that H is a subgroup of B (which is also a subgroup); therefore, we have $((\tilde{\alpha}(x) + \tilde{\beta}(y)) + c') + \tilde{\beta}(y') = (\tilde{\alpha}(x) + c' + \tilde{\beta}(y)) + \tilde{\beta}(y') = \tilde{\alpha}(x) + c' = c' + \tilde{\alpha}(x)$.

2. Correspondence between $\mathscr{C}$ and $\mathscr{L}$

The next propositions carry out the correspondence between $\mathscr{C}$ and $\mathscr{L}$. Namely, we shall define two mappings: $T : \mathscr{C} \to \mathscr{L}$ and $T' : \mathscr{L} \to \mathscr{C}$ such that $(T' \circ T)(N)$ and N are isomorphic near-rings, for any $N \in \mathscr{C}$, while $(T \circ T')(L)$ and L are isomorphic loops, for any $L \in \mathscr{L}$. We call such a correspondence a *Mal'cev's correspondence* between the classes $\mathscr{C}$ and $\mathscr{L}$. The established correspondence will be an equivalence between the formalized theories $\mathscr{I}_{\mathscr{C}}$ and $\mathscr{I}_{\mathscr{L}}$ of the two classes $\mathscr{C}$ and $\mathscr{L}$ (in the sense of [9]; see also [7]). (We note that the two classes are axiomatizable). This means that there exist two recursive mappings (algorithms) $\tilde{T} : \mathscr{I}_{\mathscr{C}} \to \mathscr{I}_{\mathscr{L}}$ and $\tilde{T} : \mathscr{I}_{\mathscr{L}} \to \mathscr{I}_{\mathscr{C}}$ such that for every closed formula $\mathbf{A} \in \mathscr{I}_{\mathscr{C}}$, $\tilde{T}(\mathbf{A})$ is a closed formula of $\mathscr{I}_{\mathscr{L}}$; $\mathbf{A}$ is true on all $N \in \mathscr{C}$ if and only if $\tilde{T}(\mathbf{A})$ is true on $T(N) \in \mathscr{L}$, and, for every closed formula $\mathbf{B}$ of $\mathscr{I}_{\mathscr{L}}$, $\tilde{T}'(\mathbf{B})$ is a closed formula of $\mathscr{I}_{\mathscr{C}}$, $\mathbf{B}$ being true on all $L \in \mathscr{L}$ if and only if $\tilde{T}'(\mathbf{B})$ is true on $T'(L) \in \mathscr{C}$.

PROPOSITION 2.1. *If* $(N, +, \cdot)$ *is a left* C-*ring from* $\mathscr{C}$, *then*: $L = N \times N \times N$, *together with the binary composition defined by*:

$$x + y = (y_1 + x_1, y_2 + x_2, y_3 + x_2 \cdot y_1 + x_3), \ \forall x = (x_1, x_2, x_3) \in L,$$

$\forall y = (y_1, y_2, y_3) \in L$, *is a loop from* $\mathscr{L}$. *The following implications hold:* $N \in \mathscr{C}_1 \rightarrow L \in \mathscr{L}_1$, $N \in \mathscr{D} \rightarrow L \in \mathscr{L}_2$, $N \in \mathscr{D}_1 \rightarrow L \in \mathscr{G}$.

Proof. Obviously, so we have a binary composition on L, with $0 = (0, 0, 0)$ as its unique two-sided zero. Now the equations $a + x = b$, $y + a = b$, have unique solutions for any $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ from L, namely: $x = (b_1 - a_1, b_2 - a_2, b_3 - a_3 - a_2 \cdot (b_1 - a_1))$ and $y = (-a_1 + b_1, -a_2 + b_2, -(-a_2 + b_2) \cdot a_1 - a_3 + b_3)$. It is easy to prove that the mappings α and β from L to L, given by:

$$\alpha(x) = (0, 0, x_2), \beta(x) = (0, 0, x_1), \quad \forall x = (x_1, x_2, x_3) \in L$$

are endomorphisms of L, with $A = \operatorname{Ker} \alpha = \{(x_1, 0, x_3) \mid x_1, x_3 \in N\}$, $B = \operatorname{Ker} \beta = \{(0, x_2, x_3) \mid x_2, x_3 \in N\}$. We can directly verify that A and B are groups with the properties asked by the axioms (ii)-(v). For instance, for any $x = (x_1, 0, x_3) \in A$, $w = (0, w_2, w_3) \in B$, $y, z \in L$, we have: $(x + y) + z = x + (y + z) = (z_1 + y_1 + x_1, z_2 + y_2, z_3 + y_2 \cdot z_1 + y_3 + z_3)$ and $(y + z) + w = y + (z + w) = (z_1 + y_1, w_2 + z_2 + y_2, w_3 + z_3 + y_2 \cdot z_1 + y_3)$, hence $B \subseteq K_\rho$, $A \subseteq K_\lambda$.

Define the functions $\tilde{\alpha} : H \rightarrow B$, $\tilde{\beta} : H \rightarrow A$, by:

$$\tilde{\alpha}(x) = (0, x_3, 0), \tilde{\beta}(x) = (x_3, 0, 0), \quad \forall x = (0, 0, x_3) \in H.$$

They are group homomorphisms and $\tilde{\alpha}(H)$, $\tilde{\beta}(H)$ satisfy the axioms (iii)-(v) (straightforward calculations). Therefore, L belongs to $\mathscr{L}$. Now if N is a strict C-ring (from $\mathscr{C}_1$), then for any $x \in H$ and $y \in L$, $x = (0, 0, x_3)$, $y = (y_1, y_2, y_3)$, we have $\tilde{\alpha}(x) + (\tilde{\alpha}(x') + y) = y$, hence $L \in \mathscr{L}_1$. If $N \in \mathscr{D}$, then $L \in \mathscr{L}_2$, since $(x + \tilde{\alpha}(y)) + z = x + (\tilde{\alpha}(y) + z)$, for any $x, z \in L$, $y \in H$. If $N \in \mathscr{D}_1$, then L is a group, and, hence L belongs to $\mathscr{G}$.

PROPOSITION 2.2. *If* $(L, +, 0)$ *is a loop from* $\mathscr{L}$, *then* H *is a near-ring from* $\mathscr{C}$ *with respect to the binary operations*:

$$x \oplus y = y + x,$$
$$x \odot y = [\tilde{\alpha}(x), \tilde{\beta}(y)], \quad \forall x, y \in H.$$

If $L \in \mathscr{L}_1$ *(resp.* $\mathscr{L}_2, \mathscr{G}$*), then* $H \in \mathscr{C}_1$ *(resp.* $\mathscr{D}, \mathscr{D}_1$*)*.

Proof. It is clear that $(H, \oplus)$ is a group [Remark 1.1], and $x \odot y \in H$, for any $x, y \in H$ [Lemma 1.1]. We have: $0 \odot y = [\tilde{\alpha}(0), \tilde{\beta}(y)] = [0, \tilde{\beta}(y)] = 0$, for any $y \in H$, by using (1.2) or (1.3). To prove the left distributivity of $\odot$ over $\oplus$, we use the following facts: (1.3), $[\tilde{\alpha}(x), \tilde{\beta}(z)] \in K_\lambda$, $[\tilde{\alpha}(x), \tilde{\beta}(z)] \in K_\lambda$, (V), $\tilde{\beta}(y') \in K_\lambda$, (1.3), $\tilde{\alpha}(x) \in K_\rho$, $\tilde{\alpha}(x) \in K_\rho$, $\tilde{\beta}(y') \in K_\lambda$, $\tilde{\beta}(y') + \tilde{\beta}(z') = \tilde{\beta}(y' + z') \in K_\lambda$, $\tilde{\beta}(z) \in K_\mu$, $(z + y)' = y' + z'$, $\tilde{\beta}(z) \in K_\mu$, and of course, the additivity of $\tilde{\alpha}$ and $\tilde{\beta}$ whenever necessary. We have: $(x \odot y) \oplus (x \odot y) = [\tilde{\alpha}(x), \tilde{\beta}(z)] + [\tilde{\alpha}(x), \tilde{\beta}(y)] = [\tilde{\alpha}(x), \tilde{\beta}(z)] + ((\tilde{\beta}(y') + \tilde{\alpha}(x)) + (\tilde{\beta}(y) + \tilde{\alpha}(x'))) = ([\tilde{\alpha}(x), \tilde{\beta}(z)] + (\tilde{\beta}(y') + \tilde{\alpha}(x))) + (\tilde{\beta}(y) + \tilde{\alpha}(x')) = (([\tilde{\alpha}(x), \tilde{\beta}(z)] + \tilde{\beta}(y')) + \tilde{\alpha}(x)) + (\tilde{\beta}(y) + \tilde{\alpha}(x')) = (\tilde{\beta}(y') + ([\tilde{\alpha}(x), \tilde{\beta}(z)] + \tilde{\alpha}(x))) + (\tilde{\beta}(y) + \tilde{\alpha}(x')) = (\tilde{\beta}(y') + (((\tilde{\beta}(z') + \tilde{\alpha}(x)) + (\tilde{\beta}(z) + \tilde{\alpha}(x'))) + \tilde{\alpha}(x))) + (\tilde{\beta}(y) + \tilde{\alpha}(x')) = (\tilde{\beta}(y') + ((\tilde{\beta}(z') + \tilde{\alpha}(x)) + \tilde{\beta}(z))) + (\tilde{\beta}(y) + \tilde{\alpha}(x')) =$

$= ((\tilde{\beta}(y') + \tilde{\beta}(z')) + (\tilde{\alpha}(x) + \tilde{\beta}(z))) + (\tilde{\beta}(y) + \tilde{\alpha}(x')) = (\tilde{\beta}(y' + z') + \tilde{\alpha}(x)) + (\tilde{\beta}(z) + {} + (\tilde{\beta}(y) + \tilde{\alpha}(x'))) = (\tilde{\beta}((z + y)') + \tilde{\alpha}(x)) + (\tilde{\beta}(z + y) + \tilde{\alpha}(x')) = [\tilde{\alpha}(x), \tilde{\beta}(z + y)] = x \odot (y \oplus z)$, for all $x, y, z \in H$. The second statement of Proposition 2.2 can be verified in the same manner.

THEOREM 2.3. (i) *There is a Mal'cev's correspondence between the classes $\mathscr{C}$ and $\mathscr{L}$.* (ii) *The theories of the two classes are equivalent.*

Proof. (i) Define $T : \mathscr{C} \to \mathscr{L}$, by $T(N) = L, \forall N \in \mathscr{C}$, as in Proposition 2.1, and $T' : \mathscr{L} \to \mathscr{C}$, by $T'(L) = H$, $\forall L \in \mathscr{L}$, as in Proposition 2.2. We have the near-ring isomorphisms, $\tau : N \to T'(T(N)), \forall N \in \mathscr{C}$, given by:

$$\tau(x) = (0, 0, x), \qquad \forall x \in N.$$

(The proof is quite simple and we omit it).

Then we construct the function $\sigma : T(T'(L)) \to L, \forall L \in \mathscr{L}$, by defining:

$$\sigma((x_1, x_2, x_3)) = \tilde{\beta}(x_1) + x_3 + \tilde{\alpha}(x), \qquad \forall (x_1, x_2, x_3) \in T(T'(L)),$$

hence $x_1, x_2, x_3 \in H \subseteq L$. Note that in the definition of σ, we can avoid using brackets, because of one of the relations: $\tilde{\beta}(x_1) \in K_\lambda$ or $\tilde{\alpha}(x_2) \in K_\rho$, which are both true. We have: $\sigma(x + y) = \tilde{\beta}(x_1 + y_1) + (x_3 + [\tilde{\alpha}(x_2), \tilde{\beta}(y_1)] + y_3) + \tilde{\alpha}(x_2 + y_2) = \tilde{\beta}(x_1) + ((\tilde{\beta}(y_1) + x_3) + {} + ([\tilde{\alpha}(x_2), \tilde{\beta}(y_1)] + y_3)) + (\tilde{\alpha}(x_2) + \tilde{\alpha}(y_2)) = (\tilde{\beta}(x_1) + x_3) + (\tilde{\beta}(y_1) + ((\tilde{\beta}(y_1') + \tilde{\alpha}(x_2)) + {} + (\tilde{\beta}(y_1) + \tilde{\alpha}(x_2'))) + y_3)) + (\tilde{\alpha}(x_2) + \tilde{\alpha}(y_2)) = ((\tilde{\beta}(x_1) + x_3 + \tilde{\alpha}(x_2)) + ((\tilde{\beta}(y_1) + (\tilde{\alpha}(x_2') + y_3))) + {} + (\tilde{\alpha}(x_2) + \tilde{\alpha}(y_2)) = \sigma(x) + (((\tilde{\beta}(y_1) + y_3) + \tilde{\alpha}(x_2')) + (\tilde{\alpha}(x_2) + \tilde{\alpha}(y_2))) = \sigma(x) + (\tilde{\beta}(y_1) + {} + y_3 + \tilde{\alpha}(y_2)) = \sigma(x) + \sigma(y), \forall x, y \in T(T'(L))$, hence σ is a loop homomorphism. Let x be an element of L, then $x_1 = \beta(x), x_2 = \alpha(x), x_3 = \tilde{\beta}(x_1') + x + \tilde{\alpha}(x_2')$ are in H (we prove it, by applying α and β to them). We have $\sigma((x_1, x_2, x_3)) = x$. Therefore σ is surjective. Since $\sigma(x) = \sigma(y)$ implies that $x_1 = y_1, x_2 = y_2$, hence $x_3 = y_3$ and $x = y$, σ is injective. Therefore σ is a loop isomorphism. Hence T and T′ define a Mal'cev's correspondence between $\mathscr{C}$ and $\mathscr{L}$.

(ii) Consider the standard formalized theories $\mathscr{I}_{\mathscr{C}}$ and $\mathscr{I}_{\mathscr{L}}$, in the sense of [9], of the classes $\mathscr{C}$ and $\mathscr{L}$. We note that the list of their primitive symbols contains, respectively, the special symbols: $\{+, \cdot, \cdot, 0\}$ for $\mathscr{I}_{\mathscr{C}}$ and $\{+, 0, \alpha(\ ), \beta(\ ), \tilde{\alpha}(\ ), \tilde{\beta}(\ ), [\ ,\ ]\}$ for $\mathscr{I}_{\mathscr{L}}$, to denote: algebraic operations, neutral elements, additive operators (as unary predicates), commutator brackets for denoting the solution of an equation (1.1). By x' we denote the element of $L \in \mathscr{L}$ which satisfies equalities $x' + x = 0 = x + x'$, for $x \in L$. We define a recursive mapping $\tilde{T} : \mathscr{I}_{\mathscr{C}} \to \mathscr{I}_{\mathscr{L}}$ thus: Let **A** be a closed formula of $\mathscr{I}_{\mathscr{C}}$. Then $\tilde{\mathbf{A}}$, obtained from **A** by replacing $x_i + x_j$ by $x_j + x_i$, 0 by 0, and $x_i \cdot x_j$ by $[\tilde{\alpha}(x_i), \tilde{\beta}(x_j)]$, is a formula of $\mathscr{I}_{\mathscr{L}}$. Now $\tilde{T}(\tilde{\mathbf{A}}) = \tilde{\mathbf{A}}^{(P)}$, where $\tilde{\mathbf{A}}^{(P)}$ is obtained by relativizing $\tilde{\mathbf{A}}$ to the predicate P, given by "$x \in \operatorname{Ker} \alpha \cap \operatorname{Ker} \beta$" [9, I. 5, p. 25]. By Proposition 2.1, we see that **A** is true on $N \in \mathscr{C}$ if and only if $\tilde{T}(\mathbf{A})$ is true on $\tilde{T}(N) \in \mathscr{L}$. For the converse, assume that every closed formula **B** of $\mathscr{I}_{\mathscr{L}}$ is under its prenex form: $\mathbf{B} = (Q_1 x_1)(Q_2 x_2), \cdots, (Q_n x_n) \mathbf{B}_1(x_1, x_2, \cdots, x_n, 0)$, where Q_i represents a quantifier and the formula $\mathbf{B}_1 \in \mathscr{I}_{\mathscr{L}}$ does not contain other quantifiers (see [4, II, § 3.5]). Construct $\tilde{T}'(\mathbf{B})$ in $\mathscr{I}_{\mathscr{C}}$ by replacing $(Q_i x_i)$ by $(Q_i x_i)(Q_i y_i)(Q_i z_i)$, $i = 1, 2, \cdots, n$, and the expressions of the form $x_i + x_j = x_k$ by $(x_j + x_i = x_k) \wedge (y_j + y_i = y_k) \wedge (z_j + y_i \cdot x_j + z_i = z_k)$. As it is obvious from the construction of T′, **B** is true on $L \in \mathscr{L}$ if and only if $\tilde{T}'(\mathbf{B})$ is true on $T'(L) \in \mathscr{C}$. Note that one must be careful with the "translations" of the formulas of $\mathscr{I}_{\mathscr{C}}$ by means of $\tilde{T}$, because the members of $\mathscr{L}$ have nonassociative additions and, therefore, one must use brackets to show the order of the additions contained in these formulas. But when we relativise to the predicate P, the associativity law holds again, and then brackets become superfluous.

It is easy to prove the following:

COROLLARY 2.4. *The restrictions of* T *and* T′ (*respectively*, $\tilde{T}$ *and* $\tilde{T}'$) *to the classes* $\mathscr{C}_1$ *and* $\mathscr{L}_1$, $\mathscr{D}$ *and* $\mathscr{L}_2$, $\mathscr{D}_1$ *and* $\mathscr{G}$ (*respectively to their formalized theories*) *define a Mal'cev's correspondence* (*an equivalence*) *between them.*

The last statement (about the correspondence between $\mathscr{D}_1$ and $\mathscr{G}$) is the main result of our previous paper [8].

We note now the functorial aspect of the established correspondence:

THEOREM 2.5. *The categories* $\tilde{\mathscr{C}}$ (*respectively* $\tilde{\mathscr{C}}_1$, $\tilde{\mathscr{D}}$, $\tilde{\mathscr{D}}_1$) *and* $\tilde{\mathscr{L}}$ (*respectively* $\tilde{\mathscr{L}}_1$, $\tilde{\mathscr{L}}_2$, $\tilde{\mathscr{G}}$) *are equivalent* (see [6, II]).

We give only the representative functors between these categories. First, we have: $F : \tilde{\mathscr{C}} \to \tilde{\mathscr{L}}$ given by:

$$F(N) = T(N), \qquad \forall N \in \mathscr{C} \text{ (Proposition 2.3)},$$

$$F(\eta) = (\eta, \eta, \eta), \qquad \forall \eta \in \mathrm{Hom}_{\tilde{\mathscr{C}}}(N, N'),$$

with $F(\eta)(x) = (\eta(x_1), \eta(x_2), \eta(x_3))$, $\forall x = (x_1, x_2, x_3) \in L$. Secondly, we have: $G : \tilde{\mathscr{L}} \to \tilde{\mathscr{C}}$, given by:

$$G(L) = T'(L), \qquad \forall L \in \mathscr{L} \text{ (Proposition 2.3)},$$

$$G(\varphi) = \varphi|_H, \qquad \forall \varphi \in \mathrm{Hom}_{\tilde{\mathscr{L}}}(L, L'),$$

with $\varphi|_H(x) = \varphi(x)$, $\forall x \in H \subseteq L$.

Let us finally remark that a Mal'cev's correspondence can be considered for the general situation of the class of left nonassociative near-rings and a special class of quasigroups. We shall handle it in a subsequent paper.

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