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Unsteady Magnetoaerodynamic Forces on an Oscillating Circular Cylindrical Shell of Finite Length. Part II: Transient solution

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Magnetofluidodinamica. — *Unsteady Magnetoaerodynamic Forces on an Oscillating Circular Cylindrical Shell of Finite Length.* Part II: *Transient solution.* Nota di LIVIU LIBRESCU, presentata [*] dal Socio C. FERRARI.

RIASSUNTO. — Questo lavoro, che costituisce il seguito di [1], è dedicato alla determinazione analitica delle forze magnetoaerodinamiche transienti su un pannello cilindrico circolare di lunghezza finita, immerso in una corrente di gas conduttore ideale, supersonico, in presenza di un campo magnetico.

1. Within the investigations concerning the motion of a thin elastic body in an electrically conduction gas flow, there are two problems of a major practical interest. These are the dynamical structural instability (known also as aero-magneto-flutter, see [2]) and the *dynamic response*, consisting in determining the stresses and displacements induced in the structure, in the context of the magneto-aeroelastic interaction and the existence of an *external* pressure field.

The last two parts of the work are devoted to this second problem, being directed towards two ends. The former of these (constituting the content of Part II) consists in determining the magneto-aerodynamic (M—A) forces. In this connexion it is worth remarking that in contrast to the analysis undertaken in [1] (in which the derived forces, involving a simple harmonic time-variation, are appropriate to flutter analysis), in the present instance, the more general case of arbitrary time-dependence of the system motion is to be considered [1].

The latter end (forming the content of Part III) consists in the exhibition of an analytical framework allowing the determination of the panel-response characteristics.

2. Let us consider a circular cylindrical thin shell of finite length l placed in an external, supersonic, ideally conducting gas flow, a magnetic field (with $\mathbf{H} \parallel \mathbf{U}$) being present. In order to derive the M—A forces, we adopt as valid the physical assumptions exhibited in [1] concerning the electro-conducting gas flow. Consequently, we start from the M—A field equations (I.1) to which we adjoin Eq. (I.2).

(*) Nella seduta del 10 dicembre 1977.

(1) The notations used throughout this note without any special mention, maintain the same significance as in Part I of the work (denoted by [1]). For the sake of brevity the equations comprised in [1] to which we shall constantly refer, will be denoted by the prefix I.

Assuming an arbitrary time dependence of the system motion, we consider the following representation of the deflection

$$(1)\qquad w\,(x_1\,,x_2\,,t) = \mathrm{W}\,(x_1\,,t)\cos n\theta$$

and of the remaining unknown functions

$$(2)\qquad \begin{aligned} f\,(x_1\,,x_2\,,x_3\,,t) &= \tilde{f}\,(x_1\,,x_3\,,t)\cos n\theta\,,\\ g\,(x_1\,,x_2\,,x_3\,,t) &= \tilde{g}\,(x_1\,,x_3\,,t)\sin n\theta\,,\end{aligned}$$

where $f\,(x_i\,,t)$ and $g\,(x_i\,,t)$ denote generically one of the functions $v_1(x_i\,,t)$, $v_3(x_i\,,t)$, $h_1(x_i\,,t)$, $h_3(x_i\,,t)$ and $v_2(x_i\,,t)$, $h_2(x_i\,,t)$, respectively, while $(x_1\,,x_2\,,x_3)$, as in [1], denotes a cylindrical coordinate system. Further, the problem will be treated by employing in the field equations both a Laplace transform ($\mathscr{L}$. T.) with respect to $\bar{x}_1$ and a Fourier transform ($\mathscr{F}$. T.), with respect to $\bar{t}$, where $\bar{x}_i \equiv x_i/l\ (i = 1\,,2)$; $\bar{x}_3 \equiv x_3/\mathrm{R}$ and $\bar{t} \equiv t\mathrm{U}/l$ denote the spatial and temporal dimensionless coordinates, respectively.

Let us define the required transforms as:

$$(3)\qquad f^{\mathscr{L}\mathscr{F}}(s\,,\bar{x}_3\,;p) \equiv \mathscr{L}\mathscr{F}\,\{\tilde{f}\} = \int\limits_0^{\infty}\int\limits_{-\infty}^{+\infty} \tilde{f}\,(\bar{x}_1\,,\bar{x}_3\,;\bar{t})\exp\left(-s\bar{x}_1 - jp\bar{t}\right)\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{t}\,, \qquad (j = (-1)^{1/2})$$

$\tilde{f}$ being one of the functions entering in the field equations, while s and p are dimensionless variables in $\mathscr{L}$. T. and $\mathscr{F}$. T., respectively.

Applying (3) to Eqs. (I.1) and (I.2), invoking the continuity condition of disturbances at $\bar{x}_1 = 0^{\pm}$ (see [1]), as well as the finiteness one for $|\bar{t}| \to \infty$, and following the same steps as in [1], all these yield a governing equation for $\rho^{\mathscr{L}\mathscr{F}}$ similar to that for $\mathring{\rho}$ (see Eq. (I.8); furthermore, one gets the expression for $\mathrm{P}^{\mathscr{L}\mathscr{F}}_{|\bar{x}_3=1} = (p_+ - p_- - t^{33}_{+})^{\mathscr{L}\mathscr{F}}_{|\bar{x}_3=1}$, as under

$$(4)\qquad \mathrm{P}^{\mathscr{L}\mathscr{F}}_{|\bar{x}_3=1} = \mathring{a}_0^2\,\rho_0\,\frac{\mathrm{R}}{l^2}\,(\breve{\mu}^2 - \lambda^2 s^2)\,\psi\,(\breve{\zeta})\,\mathrm{W}^{\mathscr{L}\mathscr{F}}(s\,,p)\,;\ (\psi\,(\breve{\zeta}) \equiv \mathrm{K}_n\,(\breve{\zeta})/(\breve{\zeta}\mathrm{K}'_n\,(\breve{\zeta})))$$

which is similar in form to Eq. (I.13), where $\mathring{\mathrm{W}}\,(s)$ is to be replaced by $\mathrm{W}^{\mathscr{L}\mathscr{F}}(s\,,p)\ (\equiv \mathscr{L}\mathscr{F}\,\{\mathrm{W}\,(\bar{x}_1\,,\bar{t})\})$, $\breve{\mu}$ being expressed now by $\breve{\mu} = \mathrm{M}\,(jp + s)$.

At this point, in order to obtain explicitely $\mathrm{P}_{|\bar{x}_3=1}$, use will be made of the same asymptotic evaluations i) and ii) of $\psi\,(\breve{\zeta})$, given by Eqs. (I.14) and (I.18), respectively.

3. Using in (4) the evaluation i) of $\psi\,(\breve{\zeta})$, and taking the inversion into the spatial domain, we obtain

$$(5)\qquad \mathrm{P}^{\mathscr{F}}_{|\bar{x}_3=1} = \mathscr{C}\left[c_1\,\mathrm{K}\,(\bar{x}_1\,,p)\,\frac{\partial \mathrm{W}^{\mathscr{F}}(\bar{x}_1\,,p)}{\partial \bar{x}_1}\bigg|_{\bar{x}_1=0} + \int\limits_0^{\bar{x}_1}\mathrm{K}\,(\bar{x}_1 - \bar{\xi}_1\,;p)\,\times\right.$$
$$\left.\times\left(c_1\,\frac{\partial^2}{\partial\bar{\xi}_1^2} + 2\,j\mathrm{M}^2 p\,\frac{\partial}{\partial\bar{\xi}_1} - \mathrm{M}^2 p^2\right)\mathrm{W}^{\mathscr{F}}\,(\bar{\xi}_1\,,p)\,\mathrm{d}\bar{\xi}_1\right],$$

the kernel function $K(\bar{x}_1, p)$ being defined in terms of dimensionless quantities $\bar{\omega}, \mathscr{C}_1, c_1, \Gamma$ as in Eq. (I.16), the only exception referring to $\check{\omega}$ which is to be replaced therein by p.

In order to obtain the original, i.e. $P_{|\bar{x}_3=1}$, the inverse $\mathscr{F}$. T. of (5) will be taken; therefore, we multiplicate both members of (5) by $(2\pi)^{-1} \exp(jp\bar{t})$ and integrate them from $-\infty$ to $+\infty$. Further, the convolution theorem (in the context of $\mathscr{F}$. T.) as under the modified form

$$(6) \quad \iint_{-\infty}^{+\infty} f_1(\bar{\tau}) f_2^{\mathscr{F}}(p) e^{jp(\bar{t}-\bar{\tau})} \mathrm{d}p \, \mathrm{d}\bar{\tau} = \int_{-\infty}^{+\infty} f_1^{\mathscr{F}}(p) f_2^{\mathscr{F}}(p) e^{jp\bar{t}} \mathrm{d}p \, , \left(f_i^{\mathscr{F}}(p) \equiv \mathscr{F}\{f_i(\bar{t})\}\right)$$

will be used. All that yields the M—A pressure expression

$$(7) \quad P_{|\bar{x}_3=1} = \mathscr{C}\left[c_1 \int_{-\infty}^{+\infty} \tilde{K}(\bar{x}_1, \bar{t}-\bar{\tau}) \left.\frac{\partial W(\bar{x}_1, \bar{\tau})}{\partial \bar{x}_1}\right|_{\bar{x}_1=0} \mathrm{d}\bar{\tau} + \right.$$

$$+ \int_0^{\bar{x}_1} \int_{-\infty}^{+\infty} \tilde{K}(\bar{x}_1 - \bar{\xi}_1 ; \bar{t} - \bar{\tau}) \left(c_1 \frac{\partial^2}{\partial \bar{\xi}_1^2} + 2\,M^2 \frac{\partial^2}{\partial \bar{\xi}_1 \partial \bar{\tau}} + M^2 \frac{\partial^2}{\partial \bar{\tau}^2}\right) \times$$

$$\left. \times W(\bar{\xi}_1, \bar{\tau}) \, \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\bar{\tau}\right] ,$$

depending upon the instantaneous values of the panel-deflection as well as the past history of the motion, where

$$\tilde{K}(\bar{x}_1, \bar{t}-\bar{\tau}) \equiv \int_{-\infty}^{+\infty} K(\bar{x}_1, p) \, e^{jp(\bar{t}-\tau)} \, \mathrm{d}p \, .$$

4. At this point it is worth remarking that in obtaining Eq. (7) no *a priori* supposition concerning the time variation of the dependent variables has been postulated. However, for obvious physical reasons, it must be considered that prior to the beginning of motion, i.e. for the system at rest (defined over the interval $-\infty < \bar{t} \leq \bar{t}_0$ of the $\bar{t}$-axis), all the perturbations are equal to zero, being different from zero only on the interval $\bar{t}_0 \leq \bar{t} < \infty$, where $\bar{t}_0$ denotes the first moment of motion. Consequently, we shall define the *generalized perturbations*

$$(8) \qquad \tilde{\bar{f}}(\bar{x}_1, \bar{x}_3 ; \bar{t}) = Y(\bar{t}-\bar{t}_0) \, \bar{f}(\bar{x}_1, \bar{x}_3 ; \bar{t}) \, ,$$

and the *generalized deflection*

$$(9) \qquad \tilde{W}(\bar{x}_1, \bar{t}) = Y(\bar{t}-\bar{t}_0) \, W(\bar{x}_1, \bar{t}) \, ,$$

where $Y(\bar{t}-\bar{t}_0)$ denotes the Heaviside distribution defined by: $Y(\bar{t}-\bar{t}_0) = 1$, $t > t_0$; $Y(\bar{t}-\bar{t}_0) = 0$, $\bar{t} \leq \bar{t}_0$. Therefore, extended by zero for $\bar{t} < \bar{t}_0$, the above mentioned functions become, in terms of Eqs. (8) and (9), defined over the entire temporal axis. Considering $W(\bar{x}_1, \bar{t})$ as a separable function, i.e.

$$W(\bar{x}_1, \bar{t}) = V(\bar{t})\, W(\bar{x}_1), \tag{10}$$

and having in mind that Eq. (7) expresses the pressure in terms of the generalized deflection, we shall use therein besides (9) and (10), the relations

$$\begin{gathered} Y^{(1)}(\bar{t}-\bar{t}_0) = \delta(\bar{t}-\bar{t}_0) \quad ; \quad V(\bar{t})\,\delta(\bar{t}-\bar{t}_0) = V(\bar{t}_0)\,\delta(\bar{t}-\bar{t}_0)\,; \\ V(\bar{t})\,\delta^{(1)}(\bar{t}-\bar{t}_0) = V(\bar{t}_0)\,\delta^{(1)}(\bar{t}-\bar{t}_0) - V^{(1)}(\bar{t}_0)\,\delta(\bar{t}-\bar{t}_0)\,, \end{gathered} \tag{11}$$

which are well-known in distribution (or generalized function) theory—see e.g. [3]—, where $\delta(\bar{t})$ is Dirac's distribution, while $f^{(1)}$ denotes the first order time derivative either of the distributions δ or Y, or of the assumed infinitely, smooth function $V(\bar{t})$. Moreover, taking into account that the support of $Y(\bar{t}-\bar{t}_0)\,V(\bar{t})$ is contained in $\bar{t}_0 \leq \bar{t} < \infty$ and supposing that supp $\tilde{K}(\bar{x}_1; \bar{t}-\bar{t}_0) \subset \{\bar{t}\,; \bar{t}_0 \leq \bar{t} < \infty\}$, all that leads finally to the M—A pressure expressible as

$$P(\bar{x}_1, \bar{x}_3 = 1\,; \bar{t}) = \mathscr{A}(w)\,, \tag{12$_1$}$$

where $\mathscr{A}$ is the M—A operator which under zero initial conditions reads

$$\begin{aligned} \mathscr{A}(\;) = \mathscr{C}\Bigg[& c_1 \int_{\bar{t}_0}^{\bar{t}} \tilde{K}(\bar{x}_1, \bar{t}-\bar{\tau})\, \frac{\partial(\;)}{\partial \bar{x}_1}\bigg|_{\bar{x}_1=0} d\bar{\tau} + \\ & + \int_0^{\bar{x}_1}\int_{\bar{t}_0}^{\bar{t}} \tilde{K}(\bar{x}_1-\bar{\xi}_1\,; \bar{t}-\bar{\tau}) \left(c_1 \frac{\partial^2(\;)}{\partial \bar{\xi}_1^2} + 2\,M^2 \frac{\partial^2(\;)}{\partial \bar{\xi}_1\, \partial \bar{\tau}} + M^2 \frac{\partial^2(\;)}{\partial \bar{\tau}^2} \right) d\bar{\xi}_1\, d\bar{\tau} \Bigg]. \end{aligned} \tag{12$_2$}$$

The presence of spatial and temporal memory effects is to be remarked in $(12)_2$; they are expressed through the dependence of the pressure at a particular point $\bar{x}_1$ $(0 \leq \bar{x}_1 \leq 1)$ and at a particular moment $\bar{t}$ $(\bar{t}_0 \leq \bar{t} < \infty)$ upon the panel deflection at all points $0 \leq \bar{\xi}_1 \leq \bar{x}_1$ and at all previous times $\bar{t}_0 \leq \bar{\tau} \leq \bar{t}$. In this way, the physical system (whose mathematical counterpart lies in Eq. (12) meets the so called *causality principle* (see e.g. [4]).

5. The use from the very beginning of the distributional methods would allow us to get directly (12). However, our present approach to the problem has led to a more general result consisting in Eq. (7), which, besides the instance (12), allows us to obtain the pressure based on the simple harmonic time-variation. Thus, inserting in (7) W as expressed by $W(\bar{x}_1, \bar{t}) = W(\bar{x}_1)\exp(j\bar{\omega}\bar{t})$, where $\bar{\omega}$ denotes the dimensionless frequency, and further,

invoking the well-known expression of Dirac's distribution, i.e. $\delta(\bar{t}) = (2\pi)^{-1}\int\limits_{-\infty}^{+\infty} \exp(j\omega\bar{t})\, d\omega$ and having in view the role of unity played by the δ-distribution in the convolution process, all that yields Eq. (I.15), derived in an *ad hoc* manner in [1].

6. Within the evaluation ii) of $\psi(\breve{\zeta})$, from (4) we get the M—A pressure as expressed by

$$(13)\qquad P_{|\bar{x}_3=1} = -\frac{U^2 \rho_0 R}{nl^2}\left[(1-\lambda^2) w'' + 2\dot{w}' + \ddot{w}\right], \qquad \forall\lambda^2 > 0,$$
$$((\dot{\ })' \equiv \partial^2(\ )/\partial\bar{x}_1\,\partial\bar{t})$$

where no *a priori* assumption concerning the time-dependence of the system motion has been made. At this point the concept of *generalized deflection* is to be taken again. Therefore, using (9), (11) and (1), expanding $W(\bar{x}_1, \bar{t})$ in terms of modal functions of the panel as

$$(14)\qquad W(\bar{x}_1, \bar{t}) = \sum_{q=1}^{N} V_q(\bar{t})\, W_q(\bar{x}_1),$$

and invoking the homogeneous initial conditions, from (13) we get (gas index-not as an exponent!)

$$(15)\qquad P(\bar{x}_1, \bar{x}_2, \bar{x}_3 = 1; \bar{t}) = Y(\bar{t}-\bar{t}_0)\sum_{q=1}^{N} P_q(\bar{x}_1, \bar{t}) \cos n\theta,$$

where

$$(16)\qquad P_q(\bar{x}_1, \bar{t}) = -\frac{U^2 \rho_0 R}{nl^2}\left[(1-\lambda^2) V_q W_q'' + 2\dot{V}_q W_q' + \ddot{V}_q W_q\right],$$

denotes the pressure due to the mode $W_q(\bar{x}_1)$, N being the number of modes considered in the analysis.

For the sake of comparison we shall now derive the (nondimensional) generalized forces Q_{qp} as defined by $Q_{qp} = (\rho_0 U^2)^{-1}\int\limits_0^1 P_q(\bar{x}_1, \bar{t})\, W_p(\bar{x}_1)\, d\bar{x}_1$; in conjunction with (16) they transcribe as

$$(17)\qquad Q_{qp} = -\frac{R}{nl}\int_0^1 \left[(1-\lambda^2) v_q W_q'' W_p + 2\dot{v}_q W_q' W_p + \ddot{v}_q W_q W_p\right] d\bar{x}_1, \qquad \bar{t} \geq \bar{t}_0$$

where $v_q(\bar{t}) \equiv V_q(\bar{t})/l$.

Specialization of (17) for the classical case yields the results derived in [5], while for harmonic time-dependence, Eq. (13) goes into (I.19). For simply-supported edge conditions (for which $W_q(\bar{x}_1) = \sin q\pi\bar{x}_1$), Eq. (17)

transforms as

$$(18) \qquad Q_{qp} = -\frac{R}{2\,nl}\begin{cases} \ddot{v}_q - (1-\lambda^2)\,q^2\,\pi^2\,v_q\,, & (q = p) \\ \dfrac{4\,pq\,(1-(-1)^{q+p})}{p^2 - q^2}\,\dot{v}_q\,, & (q \neq p) \end{cases} \qquad (\bar{t} \geq \bar{t}_0)$$

wherefrom, for $v_q(\bar{t}) \equiv v_0 = \text{const.}$, one obtains

$$(19) \qquad Q_{qp} = (2\,nl)^{-1}\,Rq^2\,\pi^2\,v_0\,(1-\lambda^2)\,\delta_{qp}\,, \qquad (\bar{t} \geq \bar{t}_0)$$

δ_{qp} denoting Kronecker's symbol. As it may easily be seen from (18), the elements Q_{qp} of the matrix $Q \equiv (Q_{qp})$ satisfy the reciprocity relation $Q_{qp} = (-1)^{q+p}\,Q_{pq}$, whereas (19) shows that Q is a diagonal one.

The next part of the work (Part III) will deal with the exhibition of an analytical framework allowing the determination of the panel-response characteristics.

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