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The prolongations of G-spaces

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Geometria differenziale. — *The prolongations of* G-*spaces*. Nota di ALEXANDRU C. NEAGU, presentata (*) dal Socio E. MARTINELLI a nome del compianto Socio B. SEGRE.

RIASSUNTO. — In questo articolo, si ottengono i prolungamenti d'ordine r di un G–spazio differenziabile, nel quale tutte le orbite sono dello stesso tipo, G essendo un gruppo di Lie compatto.

In this paper we construct the holonomic prolongations of order r of a differentiable G-space with G a compact Lie group and all orbits of the same type (see [1]). The prolongation of order r is a G^r_m-space, where G^r_m is the semi-direct product $L^r_m \overline{\times} T^r_m G$ (see [3], [4]). As an example we consider a proper and free action of G on a manifold.

Let X be a differentiable G-space with all orbits of the same type G/H [1]. It is well known that the canonical projection of X on the orbit space $X/G = X^*$ defines a fibre bundle structure with the fibre G/H and the structure group $N/H = K$, where N is the normalizer of H in G [1].

Let $X^{(H)}$ be the fixed point space of H, then X is identified with the twisted products $X^{(H)} \times_H G$ and $X^{(H)} \times_K (G/H)$, respectively and $X^{(H)}$ is a principal fibre bundle over X^* with the structure group K (see [1]). Let Ψ be the set $\{(\varphi , \sigma)\}$ where $\varphi : U \subset R^m \to X^*$ is a local diffeomorphism about $o \in R^m$ and $\sigma : \varphi(U) \subset X^* \to X$ a local differentiable cross section over $\varphi(U)$ of the bundle $X \to X^*$. Let $\mathscr{C}^r X$ be the set $\{(j^r_0 \varphi , j^r_{\varphi(0)} \sigma)/(\varphi , \sigma) \in \Psi\}$ where $j^r_0 \varphi$ (respectively $j^r_{\varphi(0)} \sigma$) is the r-jet of φ (respectively σ) in $o \in R^m$ (respectively $\varphi(o) \in X^*$). We observe that $j^r_0 \varphi \in H^r(X^*)$ and $j^r_{\varphi(0)} \sigma \in J^r X$, where $H^r(X^*)$ is the principal fibre bundle of r-frames tangent of X^* and $J^r X$ is the fibre bundle of r-jets of the local cross sections of the fibre bundle $X \to X^*$ (see [4]). It is obvious that $\mathscr{C}^r X$ is identified with the fibre product $H^r(X^*) \times \times_{X^*} J^r X$ and hence $\mathscr{C}^r X$ is a differentianle fibre bundle.

If X is the right G-space $R^m \times G$ with the action

$$((u , a) , g) \in (R^m \times G) \times G \to (u , g \cdot a) \in R^m \times G$$

and we consider the set $\overline{\Psi} = \{(\overline{\varphi} , \overline{\sigma})\}$, where $\overline{\varphi} : U \subset R^m \to R^m$ is a local diffeomorphism such that $\overline{\varphi}(o) = o$ and $\overline{\sigma} : \overline{\varphi}(U) \to G$ is a differentiable map, then the set $\{(j^r_0 \overline{\varphi} , j^r_0 \overline{\sigma})/(\overline{\varphi} , \overline{\sigma}) \in \overline{\Psi}\}$ is identified with the semi-direct product $L^r_m \overline{\times} T^r_m G = G^r_m$ where L^r_m is the structure group of $H^r(X^*)$ and

(*) Nella seduta del 18 novembre 1977.

$T^r_m G$ is the group of m^r-velocity on G (see [3], [4]). The G^r_m has a group structure with the product

$$\bar{h}_2 \cdot \bar{h}_1 = (\bar{y}_2, \bar{S}_2) \cdot (\bar{y}_1, \bar{S}_1) = (\bar{y}_2 \cdot \bar{y}_1, (\bar{S}_2 \cdot \bar{y}_1) \cdot \bar{S}_1) \tag{1}$$

where $\bar{h}_i = (\bar{y}_i, S_i) = (j^r_0 {}_i, j^r_0 {}_i)$, $(i = 1, 2)$ (see [3]).

Analogously, let $R^m \times (H \backslash G)$ be the right G-space (where H is a closed subgroup of G) with the action:

$$((u, H \cdot a), g) \in (R^m \times (H \backslash G)) \times G \to (u, H \cdot g \cdot a) \in R^m \times (H \backslash G)$$

and we consider the set $\tilde{\Psi} = \{(\tilde{\varphi}, \tilde{\sigma})\}$, where $\tilde{\varphi} : U \subset R^m \to R^m$ is a local diffeomorphism of R^m such that $\tilde{\varphi}(0) = 0$ and $\tilde{\sigma} : \tilde{\varphi}(U) \to H \backslash G$ is a local differentiable map. Let F^r_m be the set $\{(j^r_0 \tilde{\varphi}, j^r_0 \tilde{\sigma}) / (\tilde{\varphi}, \tilde{\sigma}) \in \tilde{\Psi}\}$. It is clear that F^r_m is canonical identified with $L^r_m \times T^r_m(H \backslash G)$, where $T^r_m(H \backslash G)$ is the space of m^r-velocity on $H \backslash G$.

THEOREM 1. *F^r_m is canonically identified with the space $T^r_m H \backslash G^r_m$ where $T^r_m H$ is the group of m^r-velocity on H (subgroup of $T^r_m G$ and hence G^r_m).*

Proof. $(j^r_0 \bar{\varphi}, T^r_m H \cdot j^r_0 \bar{\sigma}) = T^r_m H \cdot (j^r_0 \bar{\varphi}, j^r_0 \bar{\sigma})$ implies that $L^r_m \times \times (T^r_m H \backslash T G^m_r) \subset T^r_m H \backslash G^r_m$. The opposite inclusion follows from:

$$j^r_0 \bar{\sigma}_1 \cdot (j^r_0 \bar{\varphi}, j^r_0 \bar{\sigma}) = (j^r_0 \, id_{R^m}, j^r_0 \bar{\sigma}_1) \cdot (j^r_0 \bar{\varphi}, j^r_0 \bar{\sigma}) =$$
$$= (j^r_0 \bar{\varphi}, j^r_0 (\bar{\sigma}_1 \circ \bar{\varphi}) \cdot \bar{\sigma}) \qquad \text{where} \quad j^r_0 \bar{\sigma}_1 \in T^r_m H .$$

We have the following identification $T^r_m(H \backslash G) = T^r_m H \backslash T^r_m G$. Indeed, let $\bar{h} = (\bar{y}, \bar{S}) = (j^r_0 \bar{\varphi}, j^r_0 \bar{\sigma})$ and $\tilde{h} = (\tilde{y}, \tilde{S}) = (j^r_0 \tilde{\varphi}, j^r_0 \tilde{\sigma})$ be the elements of G^r_m and F^r_m, respectively. Thus $(\tilde{\varphi} \circ \bar{\varphi}, (\tilde{\sigma} \circ \bar{\varphi}) \cdot \bar{\sigma}) \in \tilde{\Psi}$ and hence:

$$\tilde{h} \cdot \bar{h} = (\tilde{y} \cdot \bar{y}, (\tilde{S} \cdot \bar{y}) \cdot \bar{S}) .$$

This formula represents an action of G^r_m on $T^r_m(H \backslash G)$. It is easy, by a direct calculus, to prove that this action is transitively. The theorem will be proved if the isotropy group of any point $\tilde{h} \in F^r_m$ is conjugated to $T^r_m H$.

LEMMA. *The isotropy group of $\tilde{h}_0 = (\tilde{y}_0, \tilde{S}_0)$ is $T^r_m H$, where $\tilde{y}_0 = j^r_0 \, id_{R^m}$, $\tilde{S}_0 = j^r_0 \tilde{\sigma}_0$, and $\tilde{\sigma}_0 : u \in R^m \to \tilde{\sigma}_0(u) = H \cdot e \in H \backslash G$.*

Proof. If $\bar{h} = (\bar{y}, \bar{S}) = (j^r_0 \bar{\varphi}, j^r_0 \bar{\sigma})$, then:

$$\tilde{h}_0 \cdot \bar{h} = (\tilde{y}_0 \cdot \bar{y}, (\tilde{S}_0 \cdot \bar{y}) \cdot \bar{S}) = (j^r_0 \tilde{\varphi} \circ \bar{\varphi}, j^r_0 (\tilde{\sigma} \circ \bar{\varphi}) \cdot \bar{\sigma}) ,$$

where $\tilde{\varphi}_0 = id_{R^m}$. If $\bar{h} \in T^r_m H$, then $\bar{y} = j^r_0 \, id_{R^m}$ and $\bar{\sigma} : R^m \to H$, hence $(\tilde{\sigma}_0 \circ \bar{\varphi}) \cdot \bar{\sigma} : R^m \to H \backslash G$, $((\tilde{\sigma}_0 \circ \bar{\varphi}) \cdot \bar{\sigma})(u) = \tilde{\sigma}_0(\bar{\varphi}(u)) \cdot \bar{\sigma}(u) = H \cdot e$ and $(\tilde{\sigma}_0 \circ \bar{\varphi}) \cdot \bar{\sigma} = \tilde{\sigma}_0$. It follows that $\tilde{h}_0 \cdot \bar{h} = \tilde{h}_0$ and thus $T^r_m H$ is a subgroup of the isotropy group of $\tilde{h}_0$. In order to prove the converse inclusion, we estimate the equality $\tilde{h}_0 \cdot \bar{h} = \tilde{h}_0$, $\bar{h} \in G^r_m$ in the local coordinates about $e \in H$ and $e \cdot H \in H \backslash G$, respectively and find that $\bar{h} \in T^r_m H$.

THEOREM 2. *$\mathscr{C}^r X$ is a right G_m^r-space with the action*:

$$h \cdot \bar{h} = (y \cdot \bar{y}, S \cdot (\bar{S} \cdot \bar{y}^{-1} \cdot y^{-1})), \tag{2}$$

where $h = (y, S) \in \mathscr{C}^r X$ *and* $\bar{h} = (\bar{y}, \bar{S}) \in G_m^r$.

Proof. If $y = j_0^r \varphi$, $S = j_{\varphi(0)}^r \sigma$, $\bar{y} = j_0^r \bar{\varphi}$ and $\bar{S} = j_0^r \bar{\sigma}$, then $(\varphi \circ \bar{\varphi}, \sigma(\bar{\sigma} \circ \bar{\varphi}^{-1} \circ \varphi^{-1})) \in \Psi$. Indeed $\varphi \circ \bar{\varphi} : V \subset R^m \to X^*$ and $\sigma \cdot (\bar{\sigma} \circ \bar{\varphi}^{-1} \circ \varphi^{-1}) : (\varphi \circ \bar{\varphi})(V) \subset X^* \to X$. If $x^* \in (\varphi \circ \bar{\varphi})(V)$, then $(\sigma \cdot (\bar{\sigma} \circ \bar{\varphi}^{-1} \circ \varphi^{-1}))(x^*) = \sigma(x^*) \cdot \bar{\sigma}(\bar{\varphi}^{-1} \circ \varphi^{-1}(x^*))$. It follows that $\pi \circ (\sigma \cdot (\bar{\sigma} \circ \bar{\varphi}^{-1} \circ \varphi^{-1}))(x^*) = x^*$, where $\pi : X \to X^*$ is the canonical projection. Indeed, σ is a cross section of the fibre bundle $X \to X^*$, $(\bar{\sigma} \circ \bar{\varphi}^{-1} \circ \varphi^{-1})(x^*) \in G$ and the orbit of $\sigma(x^*)$ is mapped by π in x^*. Passing to r-jets, we obtain relation (2). By a direct computation and following (1) and (2), we have $h \cdot (\bar{h}_2 \cdot \bar{h}_1) = (h \cdot \bar{h}_2) \cdot \bar{h}_1$.

Let $\bar{h}_0 = (\bar{y}_0, \bar{S}_0)$ be the unit element of G_m^r. Then, (2) yelds $h \cdot \bar{h}_0 = h$.

THEOREM 3. *The orbits of the above action of G_m^r on $\mathscr{C}^r X$ coincide with the local fibres of the fibre bundle $\mathscr{C}^r X \to X^*$.*

Proof. Let $j^r : \mathscr{C}^r X \to X^*$ be the map defined by $j^r(h) = \varphi(0)$, where $h = (y, S) = (j_0^r \varphi, j_{\varphi(0)}^r \sigma)$. Let $\bar{h} = (\bar{y}, \bar{S})$ be an element of G_m^r. Then $j^r(h \cdot \bar{h}) = (\varphi \circ \bar{\varphi})(0) = \varphi(\bar{\varphi}(0)) = \varphi(0) = j^r(h)$. Let h_1, h_2 be two points on the same fibre and suppose that $h_i = (y_i, S_i) = (j_0^r \varphi_i, j_{\varphi(0)}^r \sigma_i)$. Then, from $j^r h_1 = j^r h_2$, we deduce that $\varphi_1(0) = \varphi_2(0)$. It follows that σ_1 and σ_2 are the cross sections over $\varphi_1(U) \cap \varphi_2(U) \subset X^*$, where U is an open neighborhood of $0 \in R^m$. This allows us to conclude that $\sigma_1(x^*)$ and $\sigma_2(x^*)$ are contained of the same orbit of G-space X. Let $\bar{\varphi}$ be the map $\varphi_1^{-1} \circ \varphi_2$ and $\bar{\sigma} : U \to G$ such that $\sigma_2 \circ \varphi_2 = (\sigma_1 \circ \varphi_2^{-1}) \cdot (\bar{\sigma} \circ \bar{\varphi}^{-1})$. Then, if we denote $\bar{h} = (j_0^r \bar{\varphi}, j_0^r \bar{\sigma})$, we have:

$$h_1 \cdot \bar{h} = (y_1 \cdot \bar{y}, S_1 \cdot (\bar{S} \cdot y^{-1} \cdot y_1^{-1})) = (y_2, S_2) = h_2 .$$

Remark 1. The fibre of $\mathscr{C}^r X \to X^*$ is F_m^r and hence $\mathscr{C}^r X$ is a G_m^r-space with all orbits of the same type. The orbit space $\mathscr{C}^r X / G_m^r$ coincides with X^*.

Remark 2. $\mathscr{C}^r X \to X^*$ is a fibre bundle with the structure group $\bar{K}_m^r = T_m^r H \backslash N(T_m^r H)$, where $N(T_m^r H)$ is the normalizer of $T_m^r H$ in G_m^r. The fixed points space of $T_m^r H$ (denoted by $\mathscr{C}^r X^{(T_m^r H)}$) is a principal fibre bundle over X^* with the structure group $\bar{K}_m^r$ [1].

THEOREM 4. $N(T_m^r H)$ *is canonically identified with* $N_m^r = L_m^r \overline{\times} T_m^r N$, *where* N *is the normalizer of* H *in* G.

Proof. Let $\bar{h} = (\bar{y}, \bar{S})$ and $\bar{h}_1 = (\bar{y}_0, \bar{S}_1)$ be two elements of G_m^r and $T_m^r H$, respectivelly. It follows that:

$$\bar{h} \cdot \bar{h}_1 \cdot \bar{h}^{-1} = (\bar{y}, \bar{S}) \cdot (\bar{y}_0, \bar{S}_1) \cdot (\bar{y}^{-1}, \bar{S}^{-1} \cdot \bar{y}^{-1}) = (\bar{y}_0, (\bar{S} \cdot \bar{S}_1 \cdot \bar{S}^{-1}) \cdot \bar{y}^{-1}) .$$

If $\bar{h} \in N_m^r$ then $\bar{\sigma} : R^m \to N$ and hence $\bar{\sigma} \cdot \bar{\sigma}_1 \cdot \bar{\sigma}^{-1} : R^m \to H$. Passing to r-jets, we have $\bar{h} \cdot \bar{h}_1 \cdot \bar{h}^{-1} \in T_m^r H$ and hence $N_m^r \subset N(T_m^r H)$, where $N(T_m^r H)$ is the normalizer of $T_m^r H$ in G_m^r.

If $\bar{h}\cdot\bar{h}_1\cdot\bar{h}^{-1}\in T^r_m H$ (with $h\in G^r_m$), then $j^r_0(\bar{\sigma}\cdot\bar{\sigma}_1\cdot\bar{\sigma}^{-1})\circ\bar{\varphi}^{-1}\in T^r_m H$ and hence we can choose a representative $\bar{\sigma}$ for $j^r_0\bar{\sigma}$ such that $\bar{\sigma}\cdot\bar{\sigma}_1\cdot\bar{\sigma}^{-1}$: $R^m\to H$. It results that $\bar{\sigma}: R^m\to N$ and hence $\bar{h}\in N^r_m$.

Remark 3. From the canonical identifications $T^r_m H\backslash N^r_m = T^r_m H\backslash N(T^r_m H) = L^r_m \bar{\times} T^r_m(H\backslash N) = L^r_m \bar{\times} T^r_m K$, we conclude that $\bar{K}^r_m$ coincides with K^r_m.

THEOREM 5. $(\mathcal{E}^r X)^{(T^r_m H)}$ *coincides with* $\mathcal{E}^r X^{(H)}$.

Proof. Let $\bar{h}_1 = (\bar{y}_0, \bar{S}_1)\in T^r_m H$ and $h = (y, S)\in\mathcal{E}^r X$. Then:

$$h\cdot\bar{h}_1 = (y\cdot\bar{y}_0, S\cdot(\bar{S}_1\cdot\bar{y}_0^{-1}\cdot y^{-1})) = (y, S\cdot(\bar{S}_1\cdot\bar{y}_1^{-1}))$$

If $h\in\mathcal{E}^r X^{(H)}$ then the cross section σ have its values in $X^{(H)}$. Let x^* be the image of $x\in X^{(H)}$ by the projection $X^{(H)}\to X^*$, then:

$$(\sigma\cdot(\bar{\sigma}_1\circ\varphi^{-1}))(x^*) = \sigma(x^*)\cdot\bar{\sigma}_1(\varphi^{-1}(x^*)) = \sigma(x^*)$$

and $j^r_{\varphi(0)}(\sigma\cdot(\bar{\sigma}\circ\varphi^{-1})) = j^r_{\varphi(0)}\sigma$, hence $h\in(\mathcal{E}^r X)^{(T^r_m H)}$. If $h\in(\mathcal{E}^r X)^{(T^r_m H)}$, then $j^r_{\varphi(0)}(\sigma\cdot(\bar{\sigma}_1\circ\varphi^{-1})) = j^r_{\varphi(0)}\sigma$. We deduce that there is a representative σ for the r-jets $j^r_{\varphi(0)}\sigma$ such that $\sigma:\varphi(U)\subset X^*\to X^{(H)}$ and hence $h\in\mathcal{E}^r X^{(H)}$.

THEOREM 6. *The space* $\mathcal{E}^r X$ *is canonically identified with the twist products* $\mathcal{E}^r X^{(H)}\times_{N^r_m} G^r_m$ *and* $\mathcal{E}^r X^{(H)}\times_{K^r_m}(T^r_m H\backslash G^r_m)$, *respectivelly.*

Proof. By a well known theorem of the theory of G-spaces (see [1]), we have the following identification $\mathcal{E}^r X = (\mathcal{E}^r X)^{(T^r_m H)}\times_{N^r_m} G^r_m$.

Remark 4. The map:

$$h = (j^r_0\varphi, j^r_{\varphi(0)}\sigma)\in\mathcal{E}^r X^{(H)}\to\sigma(\varphi(0))\in X^{(H)}$$

defines on $\mathcal{E}^r X^{(H)}$ a principal fibre bundle structure of the structure group $L^r_m \bar{\times} T^r_{m,\bar{e}} K$, where $T^r_{m,\bar{e}} K = \{j^r_0\bar{\sigma}/\bar{\sigma}: R^m\to K, \sigma(0) = \bar{e} = H\cdot e\}$.

EXAMPLE. Let X be a G-space with a proper and free action of G. Then $X\to X^*$ is a principal fibre bundle with the structure group G (see [1]). In this case $H = \{e\}$ and $T^r_m H$ coincides with the unit of G^r_m. If follows that all orbits of G^r_m-space $\mathcal{E}^r X$ have the same type G^r_m, hence G^r_m acts proper and free on $\mathcal{E}^r X$. It is clear that $\mathcal{E}^r X$ coincides with the prolongation of order r of a principal fibre bundle (see [4], [3]).

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