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Equazioni differenziali ordinarie. — *Further results on the existence of periodic solutions of a certain third order differential equation.* Nota di JAMES O. C. EZEILO, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore considera l'equazione $\dddot{x} + \psi(\dot{x})\ddot{x} + \varphi(x)\dot{x} + f(x) = p(t)$ con $p(t)$ funzione periodica di periodo ω, e con ipotesi, non molto restrittive, su $\psi(\dot{x})$, $\varphi(x)$, $f(x)$ dimostra l'esistenza di almeno una soluzione periodica di periodo ω in due casi.

1.

Consider the third order differential equation

$$\dddot{x} + a\ddot{x} + \varphi(x)\dot{x} + f(x) = p(t) \tag{1.1}$$

in which a is constant and φ, f, p are continuous functions depending only on the arguments shown and p is ω-periodic in t, that is $p(t+\omega) = p(t)$ for some $\omega > 0$. Let $\Phi(x) \equiv \int_0^x \varphi(\xi)\, d\xi$. There is a result in [1] by Reissig which shows that if the following conditions hold:

(i) $a \neq 0$, (ii) $|x|^{-1}|f(x)| \to 0$ as $|x| \to \infty$, (iii) $f(x)\operatorname{sgn} x \geq 0$ ($|x| \geq$]),

(iv) $|x|^{-1}|\Phi(x)| \to 0$ as $|x| \to \infty$ and (v) $\int_0^\omega p(t)\, dt = 0$,

then (1.1) has at least one ω-periodic solution. The restrictions (i) and (iv) here were removed in a subsequent paper [2] (See Appendix 3).

We propose, in the present paper, to examine the above result with the following weaker conditions on f, φ in place of Reissig's (ii) and (iv) respectively:

$$|f(x)| \leq A_1|x| + A_2, \tag{1.2}$$

$$|\Phi(x)| \leq B_1|x| + B_2, \tag{1.3}$$

for all x, where $A_i \geq 0$, $B_i \geq 0$ ($i = 1, 2$) are constants with A_1, B_1 sufficiently small. The investigation will, furthermore, be concerned with the more general equation

$$\dddot{x} + \psi(\dot{x})\ddot{x} + \varphi(x)\dot{x} + f(x) = p(t) \tag{1.4}$$

(*) Nella seduta del 10 dicembre 1977.

in which the coefficient ψ, not necessarily constant, is a continuous function depending only on $\dot{x}$: but our other main objective is to identify certain equations (1.4) for which, subject to the conditions ((iii) and (v) above):

$$f(x) \operatorname{sgn} x \geq 0 \qquad (|x| \geq 1) \tag{1.5}$$

$$\int_0^\omega p(t)\,dt = 0 \tag{1.6}$$

the use of *just one* (*only*) of (1.2) or (1.3) would suffice for the existence of an ω-periodic solution. The position is summed up more clearly in the following two theorems for (1.4) which will be proved shortly:

THEOREM 1. *Given the equation* (1.4) *suppose that* φ, f *and* p *are subject to* (1.3), (1.5) *and* (1.6) *respectively. Then there exists a constant* $\varepsilon_0 > 0$ *such that if* $B_1 \leq \varepsilon_0$, *then* (1.4) *admits of at least one* ω*-periodic solution for all arbitrary* $\psi(\dot{x})$.

Note here the absence of a restriction on ψ.

The next theorem covers the special case corresponding to $a \neq 0$ when results are specialized to (1.1).

THEOREM 2. *Given the equation* (1.4) *in which* p *is subject, as before, to* (1.6), *suppose that* f *is subject to* (1.2) *and* (1.5) *and that*

$$\psi(y) \geq \alpha > 0 \qquad \textit{for all } y \tag{1.7}$$

or, otherwise, that

$$\psi(y) \leq -\beta < 0 \qquad \textit{for all } y, \tag{1.8}$$

for some constants α, β. *Then there exists a constant* $\varepsilon_1 > 0$ *such that if* $A_1 \leq \varepsilon_1$ *then* (1.4) *admits of an* ω*-periodic solution for all arbitrary* $\varphi(x)$.

Observe that, when specialized to the case $\psi \equiv$ constant with f bounded Theorem 2 here gives a significant improvement on the results in [2], [3] and [4] for the same equation.

2.

The method of proof of either theorem will be by the Leray-Schauder technique, just as in [1] except that for our purpose it will be convenient here to consider the parameter-dependent equation in the form:

$$\dddot{x} + \mu\psi(\dot{x})\,\ddot{x} + \mu\varphi(x)\,\dot{x} + (1-\mu)\,c_1\,x + \mu f(x) = \mu p(t) \tag{2.1}$$

for dealing with Theorem 1, and in the form:

$$\dddot{x} + \{(1-\alpha)\,\mu + \mu\psi(\dot{x})\}\,\ddot{x} + \mu\varphi(x)\,\dot{x} + (1-\mu)\,c_2\,x + \mu f(x) = \mu p(t) \tag{2.2}$$

for dealing with Theorem 2 when ψ is subject to (1.7). The case when ψ is subject to (1.8) can also be handled with the same (2.2) but with α replaced by $(-\beta)$ as will be explained in § 6. Here in (2.1) c_1 is an arbitrarily chosen, but fixed positive constant. The constant c_2 in (2.2) is also positive, but its value is to be fixed (sufficiently small) to advantage later (see (6.4)).

The equations (2.1) and (2.2) reduce to the same (1.4) when $\mu = 1$ and to the constant-coefficient equations:

$$\dddot{x} + c_1 x = 0 \tag{2.3}$$

$$\dddot{x} + \alpha\ddot{x} + c_2 x = 0 \tag{2.4}$$

when $\mu = 0$. It is easily verified that neither of the auxiliary equations corresponding to (2.3) or (2.4) has a purely imaginary root. Thus it will now be sufficient, as in [1], for our proof of Theorem 1 or Theorem 2 with ψ subject to (1.7) to establish merely that there is fixed constant $D > 0$, whose magnitude is *independent of* μ, such that any ω-periodic solution $x(t)$ of (2.1) or (2.2), corresponding to $0 < \mu < 1$ satisfies:

$$|x(t)| \leq D, \; |\dot{x}(t)| \leq D \quad \text{and} \quad |\ddot{x}(t)| \leq D \qquad (\tau \leq t \leq \tau + \omega) \tag{2.5}$$

for some τ.

3. NOTATION

Let $A_3 \equiv \max_{0 \leq t \leq \omega} |p(t)|$. In what follows here the capitals $D, D_0, D_1 \cdots$ are finite positive constants whose magnitudes are independent of the parameter μ and, indeed, in the context of Theorem 1 depend only on c_1, A_3, B_2, φ, ψ and f, and, in the context of Theorem 2, on $c_2, A_3, A_2, \varphi, \psi$ and f. The D's without suffixes attached are not necessarily the same in each place of occurrence but the numbered D's: $D_0, D_1, \cdots$ retain a fixed identity throughout.

4. SOME PRELIMINARY RESULTS

As we shall be dealing extensively here with integrals such as $\int x^2 \, dt$, $\int \dot{x}^2 \, dt$, $\int \ddot{x}^2 \, dt$ taken over time intervals of length ω, we might as well note that if x is ω-periodic then $\int_\tau^{\tau+\omega} x^2 \, dt = \int_{\tau_0}^{\tau_0+\omega} x^2 \, dt$ for arbitrary τ and τ_0, since either integral equals $\int_0^\omega x^2 \, dt$ if x is ω-periodic. The same is true of $\int_\tau^{\tau+\omega} \dot{x}^2 \, dt$ and $\int_\tau^{\tau+\omega} \ddot{x}^2 \, dt$.

We shall require specially the use of the following two subsidiary results:

LEMMA 1. *If* $x = x(t)$ *is continuous and* ω*-periodic in* t *then*

$$\int_{\tau}^{\tau+\omega} \dot{x}^2 \, dt \leq \frac{1}{4} \omega^2 \pi^{-2} \int_{\tau}^{\tau+\omega} \ddot{x}^2 \, dt . \tag{4.1}$$

Proof of Lemma. Let x have the Fourier expansion:

$$x \sim \sum_{r=0}^{\infty} (a_r \cos 2 \pi\omega^{-1} rt + b_r \sin 2 \pi\omega^{-1} rt) , \tag{4.2}$$

so that $\dot{x}$ and $\ddot{x}$ in turn have the corresponding expansions:

$$\dot{x} \sim 2 \pi\omega^{-1} \sum_{r=1}^{\infty} - r \{a_r \sin (2 \pi\omega^{-1} rt) - b_r \cos (2 \pi\omega^{-1} rt)\}$$

$$\ddot{x} \sim - 4 \pi^2 \omega^{-2} \sum_{r=1}^{\infty} r^2 \{a_r \cos (2 \pi\omega^{-1} rt) + b_r \sin (2 \pi\omega^{-1} rt)\} .$$

We have, in the usual manner, from the expansion for $\dot{x}$ that

$$\int_{\tau}^{\tau+\omega} \dot{x}^2 \, dt = 2 \pi^2 \omega^{-1} \sum_{r=1}^{\infty} r^2 (a_r^2 + b_r^2) \tag{4.3}$$

and from the expansion for $\ddot{x}$ that

$$\begin{aligned} \int_{\tau}^{\tau+\omega} \ddot{x}^2 \, dt &= 8 \pi^4 \omega^{-3} \sum_{r=1}^{\infty} r^4 (a_r^2 + b_r^2) \\ &\leq 8 \pi^4 \omega^{-3} \sum_{r=1}^{\infty} r^2 (a_r^2 + b_r^2) \\ &\leq 4 \pi^2 \omega^{-2} \int_{\tau}^{\tau+\omega} \dot{x}^2 \, dt \end{aligned}$$

by (4.3), which proves (4.1).

LEMMA 2. *Let* $x = x(t)$ *be an* ω*-periodic solution of* (2.1) *or of* (2.2) *corresponding to* $0 < \mu < 1$

$$\int_{\tau}^{\tau+\omega} x^2 \, dt \leq D_0^2 + D_1^2 \int_{\tau}^{\tau+\omega} \dot{x}^2 \, dt \tag{4.4}$$

for some D_0 , D_1.

Proof of Lemma. Let x have the Fourier expansion (4.2) so that then

$$\int_{\tau}^{\tau+\omega} x^2 \, dt = a_0^2 + \sum_{r=1}^{\infty} (a_r^2 + b_r^2) \tag{4.5}$$

whether or not x is a solution of (2.1) or of (2.2).

If in particular $x(t)$ is a solution of (2.1) or of (2.2) then we have, in view of (1.6) on integrating (2.1), (2.2) that

$$\int_{0}^{\omega} \{(1-\mu)\, c_i x + \mu f(x)\} \, dt = 0 \qquad (i = 1, 2). \tag{4.6}$$

Since $c_i > 0$ ($i = 1, 2$) and f is subject to (1.5) it is clear from (4.6) with $0 < \mu < 1$ that

$$|x(\tau_0)| < 1 \quad \text{for some } \tau_0 \text{ such that } 0 \leq \tau_0 \leq \omega . \tag{4.7}$$

Now the coefficient a_0 in (4.2) is given by

$$a_0 = \omega^{-1} \int_{0}^{\omega} x(t) \, dt$$

$$= \omega^{-1} \int_{\tau_0}^{\tau_0+\omega} x(t) \, dt ,$$

since $x(t)$ is ω-periodic in t. Now

$$\int_{\tau_0}^{\tau_0+\omega} x(t) \, dt = t\dot{x}(t) \Big|_{\tau_0}^{\tau_0+\omega} - \int_{\tau_0}^{\tau_0+\omega} t\dot{x}(t) \, dt .$$

$$= \omega x(\tau_0) - \int_{\tau_0}^{\tau_0+\omega} t\dot{x}(t) \, dt ,$$

so that, by (4.7),

$$|a_0| < 1 + \omega^{-1} \int_{\tau_0}^{\tau_0+\omega} t \, |\dot{x}(t)| \, dt$$

and therefore, since $0 \leq \tau_0 \leq \omega$,

$$|a_0| \leq 1 + D \int_{\tau_0}^{\tau_0+\omega} |\dot{x}(t)| \, dt$$

$$\leq 1 + D \left(\int_{\tau_0}^{\tau_0+\omega} \dot{x}^2 \, dt \right)^{1/2}$$

by Schwarz's inequality. Hence

$$a_0^2 \leq D_2 \left(1 + \int_{\tau_0}^{\tau_0+\omega} \dot{x}^2 \, dt \right) \tag{4.8}$$

for sufficiently large D_2. As for the term under the summation sign in (4.5) it is clear by comparison with (4.3) that

$$\sum_{r=1}^{\infty} (a_r^2 + b_r^2) \leq D \int_{\tau}^{\tau+\omega} \dot{x}^2 \, dt . \tag{4.9}$$

The result (4.4) now follows on combining (4.8) and (4.9) with (4.5).

5. Proof of Theorem 1

Let now $x = x(t)$ be any ω-periodic solution of (2.1) with $0 < \mu < 1$ and φ subject to (1.3).

Define $I_0 \geq 0$, $I_1 \geq 0$, $I_2 \geq 0$ by:

$$I_0^2 = \int_0^{\omega} x^2 \, dt \quad , \quad I_1^2 = \int_0^{\omega} \dot{x}^2 \, dt \quad , \quad I_2^2 = \int_0^{\omega} \ddot{x}^2 \, dt .$$

Since

$$\int \dot{x}\dddot{x} \, dt = \dot{x}\ddot{x} - \int \ddot{x}^2 \, dt \qquad \text{and} \qquad \int \varphi(x) \dot{x}^2 \, dt = \dot{x}\Phi(x) - \int \ddot{x}\Phi(x) \, dt$$

we have, on multiplying (2.1) by $\dot{x}$ and integrating, that

$$I_2^2 + \mu \int_0^{\omega} \Phi(x) \ddot{x} \, dt = -\mu \int_0^{\omega} \dot{x} p(t) \, dt ,$$

so that, by (1.3) and since $0 < \mu\ 1$,

$$I_2^2 \leq B_1 \int_0^{\omega} |x| |\ddot{x}| \, dt + \left\{ B_2 \int_0^{\omega} |\ddot{x}| \, dt + A_3 \int_0^{\omega} |\dot{x}| \, dt \right\} \tag{5.1}$$

$$\leq B_1 I_0 I_2 + \omega^{1/2} (B_2 I_2 + A_3 I_1) ,$$

by Schwarz's inequality. But, by (4.4),

$$I_0 \leq D_0 + D_1 I_1 \tag{5.2}$$

$$\leq D_3 (1 + I_2) ,$$

by (4.1), for sufficiently large D_3. Thus (5.1) also implies that

$$I_2^2 \leq D_3 B_1 I_2^2 + (B_1 D_3 + D) I_2$$

by (4.1); and hence if B_1 is fixed, as we assume henceforth, such that

$$B_1 \leq \tfrac{1}{2} D_3^{-1}, \tag{5.3}$$

then

$$I_2^2 \leq DI_2,$$

from which it follows at once that

$$I_2^2 \leq D_4, \tag{5.4}$$

and then also, by (4.1), that

$$I_1^2 \leq D_5. \tag{5.5}$$

Now a combination of (4.7) with the identity:

$$x(t) \equiv x(\tau_0) + \int_{\tau_0}^{t} \dot{x}(s)\, ds$$

shows that

$$\max_{0 \leq t \leq \omega} |x(t)| < 1 + \int_{\tau_0}^{\tau_0+\omega} |\dot{x}(s)|\, ds$$

$$\leq 1 + \omega^{1/2} \left(\int_{\tau_0}^{\tau_0+\omega} \dot{x}^2(s)\, ds \right)^{1/2}$$

by Schwarz's inequality. Hence, by (5.5),

$$|x(t)| \leq D_6 \equiv 1 + \omega^{1/2} D_5^{1/2} \qquad (0 \leq t \leq \omega). \tag{5.6}$$

Next, since $x(0) = x(\omega)$ it is clear $\dot{x}(\tau_1) = 0$ for some $\tau_1 \in [0, \omega]$. Thus we have, as a result of the identity:

$$\dot{x}(t) = \dot{x}(\tau_1) + \int_{\tau_1}^{t} \ddot{x}(s)\, ds,$$

that

$$\max_{0 \leq t \leq \omega} |\dot{x}(t)| \leq \int_{\tau_1}^{\tau_1+\omega} |\ddot{x}(s)|\, ds$$

$$\leq \omega^{1/2} \left(\int_{\tau_1}^{\tau_1+\omega} \ddot{x}^2(s)\, ds \right)^{1/2},$$

by Schwarz's inequality, and therefore, by (5.4), that

$$|\dot{x}(t)| \leq D_7 \equiv \omega^{1/2} D_4^{1/2} \qquad (0 \leq t \leq \omega). \tag{5.7}$$

It remains now to establish the last estimate in (2.5). For this let us note from (2.1) that $\dddot{x} = Q$, where by virtue of (5.6) and (5.7) and the boundedness of p the function Q satisfies

$$|Q| \leq D_8 (|\ddot{x}| + 1) .$$

Thus if we multiply both sides of (2.1) by $\dddot{x}$ and integrate we shall obtain that

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \leq D_8 \int_{\tau}^{\tau+\omega} |\ddot{x}| \, |\dddot{x}| \, dt + D_8 \int_{\tau}^{\tau+\omega} |\dddot{x}| \, dt$$

$$\leq D \left(\int_{\tau}^{\tau+\omega} \ddot{x}^2 \, dt \right)^{1/2} \left(\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \right)^{1/2} + D \left(\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \right)^{1/2}$$

by Schwarz's inequality. Hence, by (5.4),

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \leq D \left(\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \right)^{1/2}$$

which in turn implies that

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \leq D_9 . \tag{5.8}$$

Now, since $\dot{x}(0) = \dot{x}(\omega)$ it follows that $\ddot{x}(\tau_2) = 0$ for some $\tau_2 \in [0, \omega]$. Therefore we have, from the identity:

$$\ddot{x}(t) = \ddot{x}(\tau_2) + \int_{\tau_2}^{t} \dddot{x}(s) \, ds ,$$

that

$$\max_{0 \leq t \leq \omega} |\ddot{x}(t)| \leq \omega^{1/2} \left(\int_{\tau_2}^{\tau_2+\omega} \dddot{x}^2 (s) \, ds \right)^{1/2}$$
$$\leq D$$

by (5.8).

This completes the verification of (2.5) for all ω-periodic solutions of (2.1) with $0 < \mu < 1$ and Theorem 1 now follows with $\varepsilon_0 = \frac{1}{2} D_3^{-1}$ (See (5.3)).

6. Proof of Theorem 2

We deal first with the case ψ subject to (1·7). Let then $x = x(t)$ be any ω-periodic solution of (2.2) with $0 < \mu < 1$. The whole substance of our proof, as pointed out in § 2 will be to establish (2.5) for $x(t)$. With the groundwork laid out in § 4 the pattern for the proof of (2.5) here is almost as in § 5 and we shall therefore skip any inessential details.

Indeed the main difference between our procedure here and the procedure in § 5 is in the method for estimating $\int_0^\omega \dot{x}^2 \, dt$. This time it is convenient to multiply our parameter—dependent equation (2.2) by x (not by $\dot{x}$ as in § 5) and then integrate. Since

$$\int x\dddot{x} \, dt = x\ddot{x} - \tfrac{1}{2}\dot{x}^2 \quad , \quad \int x\ddot{x} \, dt = x\dot{x} - \int \dot{x}^2 \, dt$$

$$\frac{d}{dt}\int_0^x \xi\varphi(\xi)\, d\xi = x\varphi(x)\dot{x} \quad , \quad \int x\psi(\dot{x})\,\ddot{x}\, dt = x\Psi(\dot{x}) - \int \dot{x}\Psi(\dot{x})\, dt$$

where $\Psi(y) \equiv \int_0^y \psi(\eta)\, d\eta$, and x is ω-periodic, the integration leads at once to the result:

$$(6.1) \qquad (1-\mu)\,\alpha \int_0^\omega \dot{x}_2 \, dt + \mu \int_0^\omega \dot{x}\Psi(\dot{x})\, dt =$$

$$= \int_0^\omega \{(1-\mu)\, c_2 x^2 + \mu x f(x) - \mu x p\}\, dt .$$

By (1.7) $\psi \geq \alpha$ and therefore also $y\Psi(y) \geq \alpha y^2$ for all y.

Thus the inequality (6.1), if (1.2) holds, implies that

$$(6.2) \qquad \int_0^\omega \dot{x}^2 \, dt \leq \alpha^{-1}(c_2 + A_1)\int_0^\omega x^2 \, dt + D\int_0^\omega |x|\, dt$$

$$\leq \alpha^{-1}(c_2 + A_1)\int_0^\omega x^2 \, dt + D\left(\int_0^\omega x^2 \, dt\right)^{1/2},$$

by Schwarz's inequality. By (4.4) and (5.2) this implies in turn that

$$(6.3) \qquad \int_0^\omega \dot{x}^2 \, dt \leq \alpha^{-1}(c_2 + A_1)\, D_1^2 \int_0^\omega \dot{x}^2 \, dt + D\left\{\left(\int_0^\omega \dot{x}^2 \, dt\right)^{1/2} + A_1 + 1\right\}.$$

Hence if for example c_2 and A_1 are fixed, as we assume henceforth, such that

$$(6.4) \qquad 0 < c_2 < \frac{1}{4}\alpha D_1^{-2} \quad , \quad A_1 \leq \frac{1}{4}\alpha D_1^{-2}$$

then we have from (6.3) that

$$\int_0^\omega \dot{x}^2 \, dt \leq D \left\{ \left(\int_0^\omega \dot{x}^2 \, dt \right)^{1/2} + 1 \right\}$$

which, in turn leads to (5.5) and therefore to (5.6) as in § 5.

It remains now to obtain the estimates for $|\dot{x}(t)|$ and $|\ddot{x}(t)|$ in (2.5).

The estimate for $|\dot{x}(t)|$ requires (5.4), just as in § 5, and to establish this we note that (2.2) implies that

$$\dddot{x} + \{(1 - \mu)\alpha + \mu\psi(\dot{x})\}\ddot{x} = R \tag{6.5}$$

where, because of the boundedness, just established, of $|x(t)|$ by a D, the function R satisfies

$$|R| \leq D(|\dot{x}| + 1).$$

Thus if we multiply both sides of (6.5) by $\ddot{x}$ and integrate we shall have, since x is ω-periodic and $(1 - \mu)\alpha + \mu\psi \geq \alpha$, that

$$\alpha \int_0^\omega \ddot{x}^2 \, dt \leq D \left(\int_0^\omega |\dot{x}| \, |\ddot{x}| \, dt + \int_0^\omega |\ddot{x}| \, dt \right)$$

$$\leq \left\{ \left(\int_0^\omega \dot{x}^2 \, dt \right)^{1/2} \left(\int_0^\omega \ddot{x}^2 \, dt \right)^{1/2} + \left(\int_0^\omega \ddot{x}^2 \, dt \right)^{1/2} \right\}$$

$$\leq D \left(\int_0^\omega \ddot{x}^2 \, dt \right)^{1/2}$$

by (5.6) which has just been established for ω-periodic solutions of (2.2). Hence

$$\int_0^\omega \ddot{x}^2 \, dt \leq D$$

as before and the estimate (5.7) then follows as in § 5 for our solution x of (2.2).

With the boundedness (each by a D) of $|x(t)|$ and $|\dot{x}(t)|$ established, the estimate (5.8) can now follow, for our solution of (2.2) exactly as in § 5, and so also the boundedness of $|\ddot{x}(t)|$ by a D for arbitrary $t \in [0, \omega]$. This concludes the verification of Theorem 2 with $\varepsilon_1 = \frac{1}{4} \alpha D_1^{-2}$ (see (6.4)) when ψ is subject to (1.7).

To tackle the case ψ subject to (1.8) we had pointed out in § 2 that we should deal with the equation (2.2) with α replaced by $(-\beta)$. The effect of the replacement on the estimate for $\int_0^\omega \dot{x}^2 \, dt$ is merely to replace α^{-1} in (6.2) by β^{-1}, as is easily seen by multiplying both sides of (6.1) by (-1) and noting that $-\mu y \Psi(y) \geq \mu \beta y^2$ so that then

$$(1-\mu)\beta \int_0^\omega \dot{x}^2 \, dt - \mu \int_0^\omega \dot{x} \Psi(\dot{x}) \, dt \geq \beta \int_0^\omega \dot{x}^2 \, dt .$$

Thus the estimate (6.3) comes through here with β in place of α and the rest of the proof when ψ is subject to (1.8) can now follow from that point exactly as before.

This completes our proof of Theorem 2.

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