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On δ-perfect functions

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Analisi matematica. — *On δ-perfect functions.* Nota di TAKASHI NOIRI, presentata (*) dal Socio E. MARTINELLI a nome del compianto Socio B. SEGRE.

RIASSUNTO. — Una funzione $f: X \to Y$ viene detta *perfetta* se f è chiusa ed $f^{-1}(y)$ è compatto per ogni $y \in Y$. In [2] sono inoltre state definite e studiate le funzioni *θ-perfette*. Qui si introducono le funzioni *δ-perfette* e si mostra che, se gli spazi X ed Y sono regolari ed f è continua, le tre suddette nozioni risultano *equivalenti*.

1. INTRODUCTION

A function $f: X \to Y$ is said to be *perfect* if f is closed and $f^{-1}(y)$ is compact for each $y \in Y$. G. T. Whyburn [9] proved that a function $f: X \to Y$ is perfect if and only if for every filter base $\mathscr{F}$ on $f(X)$ converging to $y \in Y$, $f^{-1}(\mathscr{F})$ is directed toward $f^{-1}(y)$. In [4], the Author has defined *δ-perfect* functions in a way analogous to the above characterization of perfect functions. The purpose of this Note is to investigate some relations between perfect functions and δ-perfect functions, and also those between δ-perfect functions and *θ-perfect* functions introduced by R. F. Dickman, Jr. and J. R. Porter [2].

2. PRELIMINARIES

Let S be a subset of a space X. A point $x \in X$ is called a *δ-cluster* (resp. *θ-cluster*) point of S in X [8] if $S \cap \mathrm{Int}(\mathrm{Cl}(U)) \neq \varnothing$ (resp. $S \cap \mathrm{Cl}(U) \neq \varnothing$) for every open set U containing x. The set of all δ-cluster (resp. θ-cluster) points of S is called the *δ-closure* (resp. *θ-closure*) of S and is denoted by $[S]_\delta$ (resp. $[S]_\theta$). If $[S]_\delta = S$, then S is said to be *δ-closed* in X.

DEFINITION 2.1. A function $f: X \to Y$ is said to be *δ-closed* if $[f(A)]_\delta \subset f([A]_\delta)$ for every subset A of X.

A point $x \in X$ is called a *δ-cluster* point of a filter base $\mathscr{F}$ in X if $x \in \cap \{[F]_\delta \mid F \in \mathscr{F}\} = [ad]_\delta \mathscr{F}$. A filter base $\mathscr{F}$ is said to be *δ-convergent* to a point $x \in X$ if for any open set U of X containing x, there exists an $F \in \mathscr{F}$ such that $F \subset \mathrm{Int}(\mathrm{Cl}(U))$. A filter base $\mathscr{G}$ is said to be *subordinate* to a filter base $\mathscr{F}$ if for each $F \in \mathscr{F}$ there exists a $G \in \mathscr{G}$ such that $G \subset F$. A filter base $\mathscr{F}$ is said to be *δ-directed toward* $S \subset X$ if every filter base subordinate to $\mathscr{F}$ has a *δ-cluster* point in S.

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DEFINITION 2.2. A function $f: X \to Y$ is said to be *δ-perfect* if for every filter base $\mathscr{F}$ in $f(X)$ δ-converging to $y \in Y$, $f^{-1}(\mathscr{F})$ is δ-directed toward $f^{-1}(y)$.

Remark 2.3. The continuity is not assumed on δ-perfect functions.

DEFINITION 2.4. A subset S of a space X is said to be *N-closed relative to* X [1] if for every cover $\{U_\alpha \mid \alpha \in \nabla\}$ of S by open sets of X, there exists a finite subset ∇_0 of ∇ such that $S \subset \cup \{\mathrm{Int}(\mathrm{Cl}(U_\alpha)) \mid \alpha \in \nabla_0\}$.

In [4], we have obtained the following results which are used in the sequel.

THEOREM 2.5. *A function* $f: X \to Y$ *is δ-closed if and only if the image* $f(A)$ *of each δ-closed set* A *in* X *is δ-closed in* Y.

THEOREM 2.6. *A function* $f: X \to Y$ *is δ-perfect if and only if*

(a) f *is δ-closed, and*

(b) $f^{-1}(y)$ *is N-closed relative to* X *for each* $y \in Y$.

3. δ-PERFECT FUNCTIONS

A subset S of a space X is said to be *regularly open* (resp. *regularly closed*) if $\mathrm{Int}(\mathrm{Cl}(S)) = S$ (resp. $\mathrm{Cl}(\mathrm{Int}(S)) = S$). The family of regularly open sets of X forms a basis for a topology on the underlying set of X. This new space is called the *semi-regularization* of X and is denoted by X^*. If $X^* = X$, then X is said to be *semi-regular*.

DEFINITION 3.1. A space X is said to be *almost-regular* [7] if for any point $x \in X$ and any regularly closed set A not containing x, there exist disjoint open sets U and V such that $x \in U$ and $A \subset V$.

Remark 3.2. In [7], it is shown that almost-regularity and semi-regularity are independent of each other, and also almost-regularity is strictly weaker than regularity, but they are equivalent on semi-regular spaces.

LEMMA 3.3. *If a space* X *is semi-regular* (*resp. almost-regular*), *then* $[S]_\delta = \mathrm{Cl}(S)$ (*resp.* $[S]_\theta = [S]_\delta$) *for every subset* S *of* X.

Proof. It is known that $[S]_\theta \supset [S]_\delta \supset \mathrm{Cl}(S)$ for any subset S of X [8, Lemma 1]. First, suppose that X is semi-regular and let $x \in [S]_\delta$. For any open set V containing x, there exists a regularly open set U such that $x \in U \subset V$. Since $x \in [S]_\delta$, we have $\varnothing \neq U \cap S \subset V \cap S$ and hence $x \in \mathrm{Cl}(S)$. Therefore, we obtain $[S]_\delta = \mathrm{Cl}(S)$. Next, suppose that X is almost-regular and let $x \in [S]_\theta$. For any open set V containing x, there exists a regularly open set U such that $x \in U \subset \mathrm{Cl}(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$ [7, Theorem 2.2]. Since $x \in [S]_\theta$, we have $\varnothing \neq S \cap \mathrm{Cl}(U) \subset S \cap \mathrm{Int}(\mathrm{Cl}(V))$ and hence $x \in [S]_\delta$. Therefore, we obtain $[S]_\theta = [S]_\delta$.

THEOREM 3.4. *Let* X *be a space, then the identity functions of* X *onto* X* *and of* X* *onto* X *are* δ*-perfect.*

Proof. Since X* is semi-regular, by Lemma 3.3, a subset S of X is δ-closed in X if and only if it is δ-closed in X*. Therefore, these identity functions are δ-closed and hence, by Theorem 2.6, they are δ-perfect because a singleton is compact and hence N-closed relative to the space.

THEOREM 3.5. *Let* X *be a semi-regular space. If* $f : X \to Y$ *is a* δ*-perfect function, then f is perfect.*

Proof. Suppose that X is semi-regular and $f : X \to Y$ is δ-perfect. Then by Theorem 2.6, f is δ-closed and $f^{-1}(y)$ is N-closed relative to X for each $y \in Y$. Since X is semi-regular, $f^{-1}(y)$ is compact [5, Theorem 3.1]. Let F be a closed set of X. Then, by Lemma 3.3, F is δ-closed in X. Therefore, by Theorem 2.5, $f(F)$ is δ-closed and hence closed in Y. Consequently, f is closed and has compact point inverses. This shows that f is perfect.

Remark 3.6. In Theorem 3.5, the assumption " semi-regular " on X can not be replaced by " almost-regular ", as the following example shows.

Example 3.7. Let $X = \{a, b, c\}$ and $\Gamma = \{\varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$. Then (X, Γ) is an almost-regular space which is not semi-regular. The identity function $i_X (X, \Gamma) \to (X, \Gamma^*)$ is δ-perfect by Theorem 3.4. But it is not perfect because i_X is not closed.

THEOREM 3.8. *Let* Y *be a semi-regular space. If* $f : X \to Y$ *is a perfect function, then f is* δ*-perfect.*

Proof. Suppose that Y is semi-regular and $f : X \to Y$ is perfect. Then f has compact point inverses and hence $f^{-1}(y)$ is N-closed relative to X for each $y \in Y$. Let F be a δ-closed set of X. Then F is closed in X and $f(F)$ is closed in Y. Since Y is semi-regular, by Lemma 3.3, $f(F)$ is δ-closed. Therefore, by Theorem 2.6, f is δ-perfect.

COROLLARY 3.9. *Let* X *and* Y *be semi-regular spaces. A function* $f : X \to Y$ *is* δ*-perfect if and only if f is perfect.*

Proof. This is an immediate consequence of Theorem 3.5 and Theorem 3.8.

Remark 3.10. In Theorem 3.8, the assumption " semi-regular " on Y can not be replaced by " almost-regular ", as the following example shows. Moreover, the example shows that the converse to Theorem 3.5 is not always true even if X is semi-regular. Similarly, Example 3.7 shows that the converse to Theorem 3.8 is not always true even if Y is semi-regular.

Example 3.11. Let $X = \{a, b, c\}$ and Γ be the topology for X defined in Example 3.7. Let $\Gamma_1 = \{\varnothing, \{a\}, \{c\}, \{a, c\}, X\}$. Then (X, Γ_1) is a semi-regular space. The identity function $i_X : (X, \Gamma_1) \to (X, \Gamma)$ is perfect, but not δ-perfect.

4. θ-PERFECT FUNCTIONS

DEFINITION 4.1. A subset S of a space X is said to be *rigid* (resp. *quasi H-closed relative to* X) [2] if for each cover $\{U_\alpha \mid \alpha \in \nabla\}$ of S by open sets of X, there exists a finite subfamily ∇_0 of ∇ such that

$$S \subset \mathrm{Int}(\mathrm{Cl}(\cup \{U_\alpha \mid \alpha \in \nabla_0\})) \text{ (resp. } S \subset \cup \{\mathrm{Cl}(U_\alpha) \mid \alpha \in \nabla_0\}).$$

For a subset of space X, the following implications are known [2]: compact ⇒ N-closed relative to X ⇒ rigid ⇒ quasi H-closed relative to X.

DEFINITION 4.2. A function $f: X \to Y$ is said to be *θ-continuous* [3] if for each $x \in X$ and each neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$.

DEFINITION 4.3. A function $f: X \to Y$ is said to be *θ-perfect* [2] if for every filter base $\mathscr{F}$ on $f(X)$, $\mathscr{F} \rightsquigarrow y$ implies $f^{-1}(\mathscr{F}) \rightsquigarrow f^{-1}(y)$.

LEMMA 4.4. (Dickman and Porter [2]). *If a function $f: X \to Y$ satisfies*

(a) $[f(A)]_\theta \subset f([A]_\theta)$ *for each subset* A *of* X, *and*

(b) $f^{-1}(y)$ *is rigid in* X *for each* $y \in Y$,

then f is θ-perfect. Moreover, if f is θ-continuous, then the converse holds.

THEOREM 4.5. *If* X *is an almost-regular space and $f: X \to Y$ is a θ-continuous θ-perfect function, then f is δ-perfect.*

Proof. Suppose that X is almost-regular and $f: X \to Y$ is θ-continuous θ-perfect. Then, by Lemma 4.4, $f^{-1}(y)$ is rigid and hence N-closed relative to X [6, Lemma 4]. By Lemma 3.3 and Lemma 4.4, we have

$$[f(A)]_\delta \subset [f(A)]_\theta \subset f([A]_\theta) = f([A]_\delta) \qquad \text{for every subset A of X.}$$

This shows that f is δ-closed. Therefore, by Theorem 2.6, f is δ-perfect.

COROLLARY 4.6. *If* X *is a regular space, then for a θ-continuous function $f: X \to Y$ we have the following implications:*

$$f \text{ is } \theta\text{-perfect} \Rightarrow f \text{ is } \delta\text{-perfect} \Rightarrow f \text{ is perfect.}$$

Proof. This is an immediate consequence of Theorem 3.5 and Theorem 4.5.

THEOREM 4.7. *If* Y *is an almost-regular space and $f: X \to Y$ is a δ-perfect function, then f is θ-perfect.*

Proof. Suppose that Y is almost-regular and $f: X \to Y$ is δ-perfect. Then, by Theorem 2.6, f is δ-closed and $f^{-1}(y)$ is N-closed relative to

X for each $y \in Y$. Therefore, $f^{-1}(y)$ is rigid. Moreover, by Lemma 3.3, we have

$$[f(A)]_\theta = [f(A)]_\delta \subset f([A]_\delta) \subset f([A]_\theta) \qquad \text{for every subset A of X.}$$

Therefore, by Lemma 4.4, it follows that f is θ-perfect.

COROLLARY 4.8. *If* Y *is a regular space, then for any function* $f: X \to Y$ *we have the following implications*:

$$f \text{ is perfect} \Rightarrow f \text{ is } \delta\text{-perfect} \Rightarrow f \text{ is } \theta\text{-perfect}.$$

Proof. This is an immediate consequence of Theorem 3.8 and Theorem 4.7.

COROLLARY 4.9. *A θ-continuous function of an almost-regular space into an almost-regular space is δ-perfect if and only if it is θ-perfect.*

Proof. This is an immediate consequence of Theorem 4.5 and Theorem 4.7.

COROLLARY 4.10. *Let* X *and* Y *be regular spaces. Then, for a continuous function* $f: X \to Y$, *the following are equivalent*:

(a) *f is perfect.*
(b) *f is δ-perfect.*
(c) *f is θ-perfect.*

Proof. This follows from Corollary 3.9 and Corollary 4.9.

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