Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the boundary layer function in M.H.D. flows over a flat plate. Nota II

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **63** (1977), n.5, p. 385–390.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_63_5_385_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1977.

Magnetofluidodinamica. — *On the boundary layer function in M.H.D. flows over a flat plate* (*). Nota II (**) di LUCIANO DE SOCIO e PASQUALE RENNO, presentata dal Socio G. RIGHINI.

RIASSUNTO. — Con riferimento ad un problema di perturbazione singolare che si pone per una classe di moti magnetoidrodinamici viscosi, si determina la funzione di strato limite e si perviene ad una rappresentazione asintotica uniformemente valida in ogni punto del campo d'integrazione.

3. We examine now the asymptotic behaviour of the magnetohydrodynamic motion $(\boldsymbol{v}_\varepsilon, \boldsymbol{h}_\varepsilon)$ already introduced in Note I.

On this purpose, referring to the F_i's in (2.1), we first demonstrate the following:

LEMMA 3.1. *In each point of* Q, $\forall\varepsilon \in [0, 1[$ *it is*

$$(3.1)_1 \qquad F_1(z, \tau; \varepsilon) = G_1(z, \tau) - e^{-\tau - z^2/4\varepsilon^2\tau}(\pi\tau)^{-\frac{1}{2}} + \varepsilon R_1(z, \tau; \varepsilon)$$

$$(3.1)_2 \qquad F_2(z, \tau; \varepsilon) = \frac{z e^{-\tau - z^2/4\varepsilon^2\tau}(\pi\tau)^{-\frac{1}{2}}}{2\,\varepsilon\tau} + \varepsilon R_2(z, \tau; \varepsilon)$$

$$(3.1)_3 \qquad F_3(z, \tau; \varepsilon) = G_3(z, \tau) + \varepsilon R_3(z, \tau; \varepsilon)$$

with R_1, R_2 *and* R_3 *such that*

$$(3.2) \qquad \int_0^\infty |R_i(z, \tau; \varepsilon)|\, d\tau \leq \rho_i(z; \varepsilon), \qquad (i = 1, 2, 3)$$

where ρ_1, ρ_2 *and* ρ_3 *are bounded functions,* $\forall\varepsilon \geq 0$, *vanishing at* $z = 0$.

Demonstration. Noting that for $\varepsilon < 1$ then is $\gamma = 1$, $2\alpha + \gamma = \varepsilon^{-1}$, and putting

$$\varepsilon R_2 = \frac{z e^{-z^2/4\varepsilon^2\tau}(\pi\tau)^{-\frac{1}{2}}}{2\,\varepsilon\tau}\left(e^{-\delta\tau} - e^{-\tau}\right) + \frac{\xi z^2 e^{-\delta\tau}(\pi\tau)^{-\frac{1}{2}}}{4\,\tau} \cdot$$

$$\cdot \int_0^2 (\alpha u + 1)\, e^{-z^2(\alpha u+1)^2/4\tau}\,[u/(2-u)]^{\frac{1}{2}}\, I_1\{\xi z\,[u(2-u)]^{\frac{1}{2}}\}\, du,$$

(*) Lavoro eseguito nell'ambito del G.N.F.M. - C.N.R.

(**) Pervenuta all'Accademia il 14 ottobre 1977.

from (2.3) one has

$$F_2(z, \tau; \varepsilon) = \frac{ze^{-\tau}(\pi\tau)^{-\frac{1}{2}}}{2\varepsilon\tau} e^{-z^2/4\varepsilon^2\tau} + \varepsilon R_2 .$$

Also, it is

$$e^{-\delta\tau} - e^{-\tau} = \int_{\delta\tau}^{\tau} e^{-v}\, dv \leq 3\, \varepsilon\tau e^{-\delta\tau} \tag{3.3}$$

and

$$\int_0^\infty \frac{ze^{-z^2/4\varepsilon^2\tau}(\pi\tau)^{-\frac{1}{2}}}{2\varepsilon\tau}(e^{-\delta\tau} - e^{-\tau})\, d\tau \leq 3\, ze^{-z\delta^{1/2}/\varepsilon} .$$

Furthermore, from

$$(z/2\pi^{\frac{1}{2}}) \int_0^\infty e^{-\delta\tau}(\alpha u + 1)\, e^{-z^2(\alpha u+1)^2/4\tau}\, \tau^{-3/2}\, d\tau = e^{-z(\alpha u+1)\delta^{1/2}} \tag{3.4}$$

one has

$$\varepsilon \int_0^\infty |R_2(z, \tau; \varepsilon)|\, d\tau \leq 3\, ze^{-z\delta^{1/2}/\varepsilon} + (\xi z/2) \int_0^2 e^{-z(\alpha u+1)\delta^{1/2}} [u/(2-u)^{\frac{1}{2}}] \cdot \tag{3.5}$$

$$\cdot\, I_1\{\xi z\, [u(2-u)]^{\frac{1}{2}}\}\, du .$$

Taking into account (2.10) and (3.5) it follows

$$\varepsilon \int_0^\infty |R_2(z, \tau; \varepsilon)|\, d\tau \leq 3\, ze^{-z\delta^{1/2}/\varepsilon} + \varepsilon(1 - e^{-z/\varepsilon})$$

from which $(3.1)_2$ is obtained when one puts

$$\rho_2(z, \varepsilon) = (3\, z/\varepsilon)\, e^{-z\delta^{1/2}/\varepsilon} - e^{-z/\varepsilon} + 1 .$$

The demonstration of $(3.1)_1$ is somehow lengthier but straightforward.

First of all one can observe that $(3.1)_1$ can be integrated by parts to give

$$F_1(z, \tau; \varepsilon) = \frac{e^{-\delta\tau}(\pi\tau)^{-\frac{1}{2}}}{1-\varepsilon^2}(e^{-z^2/4\tau} - e^{-z^2/4\varepsilon^2\tau}) + m_1(z, \tau; \varepsilon) - n_1(z, \tau; \varepsilon) \tag{3.6}$$

with

$$(3.7)\qquad m_1 = \frac{\xi z e^{-\delta\tau}(\pi\tau)^{-\frac{1}{2}}}{1-\varepsilon^2}\int\limits_0^2 e^{-z^2(\alpha u+1)^2/4\tau}\,[u(2-u)]^{-\frac{1}{2}}\,I_1\{\xi z\,[u(2-u)]^{\frac{1}{2}}\}\,du$$

$$(3.8)\qquad n_1 = \frac{\xi z e^{-\delta\tau}(\pi\tau)^{-\frac{1}{2}}}{1-\varepsilon^2}\int\limits_0^2 e^{-z^2(\alpha u+1)^2/4\tau}\,[u/(2-u)]^{\frac{1}{2}}\,I_1\{\xi z\,[u(2-u)]^{\frac{1}{2}}\}\,du\,.$$

Substituting $v = \alpha u + 1$ in m_1 and putting

$$(3.9)\qquad \mu_1(z,\tau,v;\varepsilon) = (1+\varepsilon)^{-2}(1-\varepsilon)^{-1}e^{-\tau/(1+\varepsilon)^2}(1-\varepsilon v)^{-\frac{1}{2}}\cdot$$
$$\cdot\, I_1\{[2z/(1-\varepsilon^2)]\,[(v-1)(1-\varepsilon v)]^{\frac{1}{2}}\}$$

then

$$(3.10)\qquad \mu_1 = e^{-\tau}\,I_1\{2z(v-1)^{\frac{1}{2}}\} + \int\limits_0^{\varepsilon}\frac{\partial}{\partial y}\mu_1(z,\tau,v;y)\,dy$$

where, from (2.9), the integrand is subject to the bound

$$(3.11)\qquad \left|\frac{\partial\mu_1}{\partial y}\right| \le \frac{4\,e^{-\tau/(1+y)^2}(1-yv)^{-\frac{1}{2}}}{(1-y^2)^3}\,\cdot$$
$$\cdot\,[\tau+2+z^2v(v-1)]\,I_1\left\{\frac{2z}{1-y^2}\,[(v-1)(1-yv)]^{\frac{1}{2}}\right\}$$

and finally through (3.7), (3.9), (3.10), (3.11) one has

$$m_1 = z e^{-\tau}(\pi\tau)^{-\frac{1}{2}}\int\limits_1^{1/\varepsilon} e^{-z^2v^2/4\tau}(v-1)^{-\frac{1}{2}}\,I_1\{2z(v-1)^{\frac{1}{2}}\}\,dv + p_1(z,\tau;\varepsilon)$$

with p_1 such that

$$|p_1| = \left| z(\pi\tau)^{-\frac{1}{2}}\int\limits_1^{1/\varepsilon} e^{-z^2v^2/4\tau}(v-1)^{-\frac{1}{2}}\,dv\int\limits_0^{\varepsilon}\frac{\partial\mu_1}{\partial y}\,dy\right| \le$$

$$\le z(\pi\tau)^{-\frac{1}{2}}\int\limits_0^{\varepsilon}dy\int\limits_1^{1/y} e^{-z^2v^2/4\tau}\cdot(v-1)^{-\frac{1}{2}}\left|\frac{\partial\mu_1}{\partial y}\right|dv\,.$$

At this point, if one puts

$$(3.12)\quad \varepsilon R_1(z, \tau; \varepsilon) = (e^{-z^2/4\tau} - e^{-z^2/4\varepsilon^2\tau})(1-\varepsilon^2)^{-1}(\pi\tau)^{-\frac{1}{2}}[e^{-\delta\tau} - (1-\varepsilon^2)e^{-\tau}] -$$

$$- n_1 + p_1 - ze^{-\tau}(\pi\tau)^{-\frac{1}{2}} \int_{1/\varepsilon}^{\infty} e^{-z^2v^2/4\tau}(v-1)^{-\frac{1}{2}} I_1\{2z(v-1)^{\frac{1}{2}}\} dv$$

taking into account (2.5) the expression $(3.1)_1$ is obtained.

However, the limitation (3.2) on the remainder R_1 is still to be established. To this purpose, from the definition of p_1 and from (3.11) and (2.9) through (2.13), via simple calculations one has

$$(3.13)\qquad \int_0^{\infty} |p_1(z, \tau; \varepsilon)| d\tau \leq 8(5 + 2z)(i - \varepsilon)^{-1} \varepsilon z .$$

For the other terms in R_1, one has from (2.11)

$$(3.14)\qquad \int_0^{\infty} |n_1(z, \tau; \varepsilon)| d\tau \leq 2\varepsilon(1-\varepsilon)^{-1}(1-e^{-z/\varepsilon})$$

while (3.3) leads to

$$(3.15)\qquad \int_0^{\infty} (e^{-z^2/4\tau} - e^{-z^2/4\varepsilon^2\tau})(1-\varepsilon^2)^{-1}(\pi\tau)^{-\frac{1}{2}}[e^{-\delta\tau} - (1-\varepsilon^2)e^{-\tau}] d\tau \leq$$

$$\leq [12\varepsilon/(1-\varepsilon)](e^{-z\delta^{1/2}} - e^{-z\delta^{1/2}/\varepsilon} + z)$$

and from (2.9) it is

$$(3.16)\qquad \int_0^{\infty} ze^{-\tau}(\pi\tau)^{-\frac{1}{2}} d\tau \int_{1/\varepsilon}^{\infty} e^{-z^2u^2/4\tau}(u-1)^{-\frac{1}{2}} I_1\{2z(u-1)^{\frac{1}{2}}\} du \leq$$

$$\leq z^2 \int_{1/\varepsilon}^{\infty} e^{-zv+2z(v-1)^{1/2}} dv \leq 6ze^{-z/2\varepsilon} .$$

Therefore if one writes

$$\rho_1(z, \varepsilon) = [12/(1-\varepsilon)][1 - e^{-z/\varepsilon} + e^{-z\delta^{1/2}} - e^{-z\delta^{1/2}/\varepsilon} + (z/\varepsilon)e^{-z/2\varepsilon} + 5z + 2z^2]$$

then $\rho_1(0, \varepsilon) = 0$ and (3.2) is verified for R_1. A similar procedure enables one to prove $(3.1)_3$ and *Lemma 3.1* is demonstrated.

4. Lemma 3.1 is now applied to the solution of problem $\mathscr{A}_\varepsilon$. Let

$$b = \frac{z e^{-\tau - z^2/4\varepsilon^2\tau} (\pi\tau)^{-\frac{1}{2}}}{2\,\varepsilon\tau} * \boldsymbol{v}^* + (\pi\tau)^{-\frac{1}{2}} (e^{-\tau - z^2/4\varepsilon^2\tau}) * \boldsymbol{h}^* \tag{4.1}$$

$$\boldsymbol{r} = R_2 * \boldsymbol{v}^* - R_1 * \boldsymbol{h}^* \tag{4.2}$$

$$\boldsymbol{s} = R_3 * \boldsymbol{h}^* - [G_1 - (\pi\tau)^{-\frac{1}{2}} e^{-\tau - z^2/4\varepsilon^2\tau} + \varepsilon R_1] * \boldsymbol{v}^* . \tag{4.3}$$

Recalling the solution of $\mathscr{A}_0$ (2.7) one has

$$\boldsymbol{v}_\varepsilon = \boldsymbol{v}_0 + \boldsymbol{b} + \varepsilon \boldsymbol{r} ,$$

$$\boldsymbol{h}_\varepsilon = \boldsymbol{h}_0 + \varepsilon \boldsymbol{s} .$$

On the other hand: $\lim_{y \to 0} \{\lambda(\tau) * [(y e^{-y^2/4\tau}/2\,\tau)(\pi\tau)^{-\frac{1}{2}}]\} = \lambda(\tau)\, , \forall \tau > 0$ and taking into account *Comment 2.1* one has

$$\lim_{z \to 0} [\boldsymbol{v}_0(z\, , \tau) + \boldsymbol{b}(z\, , \tau\, ; \varepsilon)] = \boldsymbol{v}^*(\tau)\, ; \qquad \forall \tau > 0 .$$

In order to complete the demonstration of the Theorem, it is necessary to prove the relation (1.7). If $\boldsymbol{v}^*$ and $\boldsymbol{h}^*$ are bounded as $\tau \to \infty$ and one puts

$$\xi = \sup_{\tau \geq 0} |\boldsymbol{h}^*(\tau)| \quad , \quad \varphi = \sup_{\tau \geq 0} |\boldsymbol{v}^*(\tau)|$$

(3.2) leads to

$$|\boldsymbol{r}(z\, , \tau\, ; \varepsilon)| \leq \varphi \int_0^\infty |R_2(z\, , \tau\, ; \varepsilon)|\, d\tau + \xi \int_0^\infty |R_1(z\, , \tau\, ; \varepsilon)|\, d\tau \leq \xi\rho_1 + \varphi\rho_2 .$$

If one recalls (3.13) the following relation is easily obtained

$$0 \leq \int_0^\infty [G_1(z\, , \tau) - (\pi\tau)^{-\frac{1}{2}} e^{-\tau - z^2/4\varepsilon^2\tau}]\, d\tau = 1 - e^{-z/\varepsilon}$$

and for $\boldsymbol{s}$ one has

$$|\boldsymbol{s}(z\, , \tau\, ; \varepsilon)| \leq \xi\rho_3 + \varphi(1 - e^{-z/\varepsilon}) + \varepsilon\varphi\rho_1 .$$

As for $\rho_i(0\, , \varepsilon) = 0$, $(i = 1\, , 2\, , 3)$, if one puts

$$\sigma(z\, ; \varepsilon) = \xi(\rho_1 + \rho_3) + \varphi(\rho_2 + \varepsilon\rho_1 + 1 - e^{-z/\varepsilon})$$

the demonstration has been completed.

5. The function $\boldsymbol{b}(z, \tau; \varepsilon)$, as defined by (4.1), is a classic B.L. function whose contribution is appreciable only around $z = 0$. The case where $\boldsymbol{v}^*(\tau)$ and $\boldsymbol{h}^*(\tau)$ reduce to two constant vectors, $\hat{\mathbf{V}}$ and $\hat{\mathbf{H}}$, is next considered as an application of the preceding results.

It is simple to evaluate

$$\hat{\mathbf{H}} * [(\pi\tau)^{-\frac{1}{2}} e^{-\tau - z^2/4\varepsilon^2\tau}] = \tfrac{1}{2} \hat{\mathbf{H}} [e^{-z/\varepsilon} \, erfc \, (z/2 \, \varepsilon\tau^{\frac{1}{2}} - \tau^{\frac{1}{2}}) - \\ - e^{z/\varepsilon} \, erfc \, (z/2 \, \varepsilon\tau^{\frac{1}{2}} + \tau^{\frac{1}{2}})] \, ;$$

$$[z e^{-\tau - z^2/4\varepsilon^2\tau} (\pi\tau)^{-\frac{1}{2}}/2 \, \varepsilon\tau] * \hat{\mathbf{V}} = \tfrac{1}{2} \hat{\mathbf{V}} [e^{-z/\varepsilon} \, erfc \, (z/2 \, \varepsilon\tau^{\frac{1}{2}} - \tau^{\frac{1}{2}}) + \\ + e^{z/\varepsilon} \, erfc \, (z/2 \, \varepsilon\tau^{\frac{1}{2}} + \tau^{\frac{1}{2}})] \, .$$

Therefore the B.L. function $\boldsymbol{b}(z, \tau; \varepsilon)$ is given by

$$\boldsymbol{b}(z, \tau; \varepsilon) = \tfrac{1}{2} (\hat{\mathbf{H}} + \hat{\mathbf{V}}) \, e^{-z/\varepsilon} \, erfc \, (z/2 \, \varepsilon\tau^{\frac{1}{2}} - \tau^{\frac{1}{2}}) + \\ + \tfrac{1}{2} (\hat{\mathbf{V}} - \hat{\mathbf{H}}) \, e^{z/\varepsilon} \, erfc \, (z/2 \, \varepsilon\tau^{\frac{1}{2}} + \tau^{\frac{1}{2}}) \, . \tag{5.1}$$

It is now $\boldsymbol{v}_0(0, \tau) = -\mathbf{H} \, erf \, \tau^{\frac{1}{2}}$, while $\boldsymbol{b}(0, \tau; \varepsilon) = \hat{\mathbf{V}} + \hat{\mathbf{H}} \, erf \, \tau^{\frac{1}{2}}$ and one finally has

$$\boldsymbol{b}(0, \tau; \varepsilon) + \boldsymbol{v}_0(0, \tau) = \hat{\mathbf{V}} \, .$$

It is simple to show that (5.1) leads to evaluations of the B.L. thickness that correspond to the partial results already available in the literature.

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