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Non-linear high-frequency waves in Reilativistic Cosmology

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Fisica matematica. — *Non-linear high-frequency waves in Relativistic Cosmology* (*). Nota di ANGELO MARCELLO ANILE, presentata (**) dal Socio C. CATTANEO.

RIASSUNTO. — Si studia l'evoluzione di onde acustiche di ampiezza finita nel fluido cosmologico, con il metodo delle onde asintotiche di Choquet-Bruhat.

Si calcola il tempo caratteristico per la formazione dell'urto e se ne esplorano le conseguenze cosmologiche.

1. INTRODUCTION

The generation of acoustic waves in the cosmological fluid is an important step in the process of galaxy formation [1]. These waves are thought to initiate from perturbation in the " radiation era " [1], when the universe is dominated by a radiation fluid obeying the equation of state $p = 1/3\ \rho$. Because at this stage the Jeans length is of the order of the horizon size (λ_H), the perturbations with wavelength $\lambda \ll \lambda_H$ oscillate throughout the radiation era, whereas those with $\lambda > \lambda_H$ grow indefinitely. The short wavelength perturbations ($\lambda \ll \lambda_H$), which are the acoustic waves, are then severely damped by photon viscosity ([1], [2]) until the time of recombination [3]. The final masses associated with these perturbations would represent the seed fluctuations where from clusters of galaxies formed. This picture derives from an analysis of the Einstein equations in the presence of a dissipative fluid, by linearization around the unperturbed Robertson-Walker solution. However, it is well known that the non linearities of hyperbolic equations cause a distortion of the signal as well as a breakdown of the solution after a certain characteristic time [4]. In the case of hydrodynamics this is usually interpreted as the time of shock formation [5].

Therefore it is of some interest to investigate the effect of non linearities on the previous picture of the development of perturbations in an expanding universe.

The plan of the present paper is the following. First of all we study, using the method of Choquet [5], high-frequency non linear acoustic waves in the Robertson-Walker metric. Secondly, because in the case $p = 1/3\ \rho$, a special class of exact non linear solutions of the fluid dynamical equations is available [6], we check the validity of Choquet's approximation method in this particular case.

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In Section 2 we briefly review the method of Choquet [5] for treating non-linear high-frequency waves. Then we apply it to the equations describing the cosmological fluid. We study in detail the distorsion of the signal and obtain an explicit expression for the characteristic time of breaking of the solution.

In Section 3 we discuss in detail the case of a purely radiation fluid and we compare the approximate solutions with the exact ones describing simple waves [6].

Notation and Conventions

Space-time is assumed to be a pseudoriemannian differentiable manifold. The metric signature is taken to be -2. Small latin indices run from zero to 3. Greek indices run from 1 to 3. Capital latin indices run from zero to N. ∇_a represents the covariant derivative operator.

Section 2

The method of asymptotic waves for hyperbolic non linear p.d.e. has been developed by Y. Choquet [5] and generalizes to the non-linear case the well-known W. K. B. expansion technique [7], [8]. In brief, the essence of the method is the following. One considers the quasi-linear 1st order system:

$$\mathrm{L}^{\mathrm{J}}(\mathrm{U}) \equiv \mathrm{A}_{\mathrm{I}}^{\mathrm{J}a}(x, \mathrm{U}) \frac{\partial \mathrm{U}^{\mathrm{I}}}{\partial x^{a}} + b^{\mathrm{J}}(x, \mathrm{U}) = 0 \tag{1}$$

$$\mathrm{I}, \mathrm{J} = 0, 1, \cdots, \mathrm{N}; \qquad a = 0, 1, 2, 3$$

with $x \in \mathrm{V}_4$, V_4 being a 4-dimensional differential manifold.

Also one supposes that $\mathrm{U} \in \mathscr{C}^1(\mathrm{V}_4)$, $\mathrm{A}_{\mathrm{I}}^{\mathrm{J}a} \in \mathscr{C}^\infty(\mathrm{V}_4)$, $b^{\mathrm{J}} \in \mathscr{C}^\infty(\mathrm{V}_0)$. Furthermore one assumes that $\mathrm{A}_{\mathrm{I}}^{\mathrm{J}a}$ and b^{J} are analytic functions of U in the neighbourhood of some U_0. Then we can write:

$$\mathrm{A}_{\mathrm{I}}^{\mathrm{J}a}(x, \mathrm{U}) = \underset{(0)}{\mathrm{A}_{\mathrm{I}}^{\mathrm{J}a}} + \mathrm{A}_{\mathrm{IH}}^{\mathrm{J}a}(\mathrm{U}^{\mathrm{H}} - \underset{(0)}{\mathrm{U}^{\mathrm{H}}}) + \cdots \tag{2}$$

$$b^{\mathrm{J}} = \underset{(0)}{b^{\mathrm{J}}} + b_{\mathrm{H}}^{\mathrm{J}}(\mathrm{U}^{\mathrm{H}} - \underset{(0)}{\mathrm{U}^{\mathrm{H}}}) + \cdots$$

with

$$\underset{(0)}{\mathrm{A}_{\mathrm{I}}^{\mathrm{J}a}} = \mathrm{A}_{\mathrm{I}}^{\mathrm{J}a}(x, \mathrm{U}_0) \quad ; \quad \underset{(0)}{b^{\mathrm{J}}} = b^{\mathrm{J}}(x, \underset{(0)}{\mathrm{U}})$$

$$\mathrm{A}_{\mathrm{IH}}^{\mathrm{J}a} = \frac{\partial}{\partial \mathrm{U}^{\mathrm{H}}} \mathrm{A}_{\mathrm{I}}^{\mathrm{J}a}\Big|_{\underset{(0)}{\mathrm{U}}} \quad ; \quad b_{\mathrm{H}}^{\mathrm{J}} = \frac{\partial b^{\mathrm{J}}}{\partial \mathrm{U}^{\mathrm{H}}}\Big|_{\underset{(0)}{\mathrm{U}}} .$$

Let $\varphi \in \mathscr{C}^{\infty}(V_4)$, $\omega \in \mathbf{R}^+$, $\xi = \omega\varphi$. One seeks a solution to eq. (1) in the form of an asymptotic wave [5]:

$$U^I = \sum_{q=0}^{\infty} \omega^{-q} \underset{(q)}{U}^I (x , \xi) \tag{3}$$

where $\underset{(0)}{U}$ is a solution of the unperturbed equation $\underset{(0)}{A}{}_I^{Ja} \partial_a \underset{(0)}{U}^I + \underset{(0)}{b}{}^J = 0$. In order that the formal series (3) represent an asymptotic wave (1), one obtains the following necessary conditions:

i) the vanishing of the characteristic determinant,

$$A(x , l) \equiv \det (\underset{(0)}{A}{}_I^{Ja} l_a) = 0 \tag{4}$$

where $l_a = \varphi_{,a}$.

ii) let $\bar{h}_J$ and h^I the left and right eigenvectors of the matrix $\underset{(0)}{A}{}_I^{Ja} l_a$. One can write

$$\underset{(1)}{U}^I (x , \xi) = V_1 (x , \xi) h^I (x) + V^I (x) . \tag{5}$$

In the following we shall always take $V^I (x) = 0$.

Then one obtains the following propagation equation for V_1:

$$\mu A^a \partial_a V_1 + \alpha V_1 V_1' + \beta V_1 = 0 \tag{6}$$

where

$$V_1' \equiv \frac{\partial V_1}{\partial \xi} \quad ; \quad \partial_a V_1 \equiv \frac{\partial V_1}{\partial x^a}$$

$$\mu A^a \equiv h^I \bar{h}_J \underset{(0)}{A}{}_I^{Ja} \quad ; \quad \alpha \equiv A_{IL}^{Ja} l_a h^I \bar{h}_J h^L$$

$$\beta \equiv \bar{h}_J \{\underset{(0)}{A}{}_I^{Ja} \partial_a h^I + (A_{IL}^{Ja} \partial_a \underset{(0)}{U}^I + b_L^J) h^L\} .$$

Eq. (6) is a 1st order quasi linear p.d.e. and can be solved by integration of its bicharacteristic system.

iii) linear p.d.e. for the higher order terms $\underset{(q)}{U}$, $q > 1$ which are not of interest to us in this contest.

We remark that eqs. (4), (5) appear also in the theory of weakly discontinuous solutions of (1), [9].

Now we apply Choquet's method to the equations of relativistic hydrodynamics. In its general form this was also attempted in Choquet's paper. However for the cosmological applications it is more transparent to start directly from the basic equations (4), (5).

The equations of relativistic hydrodynamics are:

$$(\rho + p)\, u^a \nabla_a u^b - \gamma^{ab} \partial_a p = 0 \tag{7a}$$

$$u^a \partial_a \rho + (\rho + p) \nabla_a u^a = 0 \tag{7b}$$

where u^a is the normalized 4-velocity $u_a u^a = 1$, γ^{ab} is the projection tensor $\gamma^{ab} = g^{ab} - u^a u^b$, p and ρ are the pressure and the energy-density respectively. We also assume the adiabatic state equation:

$$p = \gamma\rho \quad , \quad 0 < \gamma \leq 1 . \tag{8}$$

The background solution of (7a, b) is taken to be the one appropriate to a Robertson-Walker universe, with zero spatial curvature, i.e. in comoving conformal coordinates.

$$ds^2 = R^2(\eta)\,[d\eta^2 - \delta_{\mu\nu}\, dx^\mu\, dx^\nu] \tag{9}$$

with

$$\underset{(0)}{u^a} = \left(\frac{1}{R}, 0, 0, 0\right) \quad ; \quad \underset{(0)}{U^I} = \begin{bmatrix} \frac{1}{R} \\ 0 \\ 0 \\ 0 \\ \underset{(0)}{p} \end{bmatrix} .$$

Eq. (4) then yields:

$$(l_0)^2 - \gamma\delta^{\mu\nu}\, l_\mu\, l_\nu = 0 \tag{10}$$

which is the well known dispersion relation for acoustic waves. Also it is easy to see that $\frac{\partial A}{\partial l^a} \neq 0$ and therefore no caustics develop and the previous formalism is perfectly adequate (absence of linear shocks, [9]). From (10) with a simple rotation we can always take $l_1 \neq 0$, $l_2 = l_3 = 0$, hence $l_0 = \pm \sqrt{\gamma}\, l$ with $l = |\, l_1 \,|$.

We take for the right and left eigenvectors of $\underset{(0)}{A_I^{Ja}}\, l_a$

$$h^I = \begin{bmatrix} 0 \\ \frac{1}{R} \\ 0 \\ 0 \\ -\left(\frac{1+\gamma}{\gamma}\right) \underset{(0)}{p}\, \frac{l_0}{p} \end{bmatrix} \quad ; \quad h_J = \left(-R, -\frac{l}{l_0} R, 0, 0, 1\right) . \tag{11}$$

If we write $K^a = \mu A^a$ we find

$$K^a = \frac{2(1+\gamma)}{R} \underset{(0)}{\rho} \left[-\frac{l}{l_0}, 1, 0, 0 \right]. \tag{12}$$

Also we find

$$\alpha = \frac{2 \underset{(0)}{\rho} (1+\gamma)(\gamma-1)}{R\gamma} l_0 \tag{13}$$

$$\beta = \frac{l_0 (1+\gamma) \underset{(0)}{\rho} \dot{R} (3\gamma - 1)}{l\gamma R^2}. \tag{14}$$

Then eq. (6) gives:

$$-\partial_0 V_1 \pm \sqrt{\gamma}\, \partial_1 V_1 + l(\gamma - 1) V_1 V_1' + \frac{3\gamma - 1}{2} \frac{\dot{R}}{R} V_1 = 0. \tag{15}$$

For $\gamma = 1$ eq. (15) is linear and we recover the exceptional waves of Lax ([5], [9]).

The bicharacteristic system of eq. (15) is:

$$\begin{aligned} &\frac{d\eta}{d\sigma} = 1 \quad ; \quad \frac{dx^1}{d\sigma} = \mp \sqrt{\gamma} \quad ; \quad \frac{dx^2}{d\sigma} = \frac{dx^3}{d\sigma} = 0 \\ &\frac{d\xi}{d\sigma} = l(1-\gamma) V_1 \quad ; \quad \frac{dV_1}{d\sigma} = \frac{3\gamma - 1}{2} \frac{\dot{R}}{R} V_1 \end{aligned} \tag{16}$$

where σ is a parameter along the bicharacteristics. We must now specify the initial conditions for the system (16). We take the hypersurface $\Sigma : \eta = \eta_E$ as initial hypersurface. We assume that:

$$\begin{aligned} &V_1 |_\Sigma = W_1(y, \mu) \quad , \quad y \in \Sigma \\ &\xi |_\Sigma = \mu . \end{aligned} \tag{17}$$

It follows:

$$\eta = \sigma + \eta_E \quad ; \quad x^1 = \mp \sqrt{\gamma}\sigma \quad ; \quad x^2 = x^3 = 0 \tag{18a}$$

$$V_1 = W_1 \left[\frac{R(\eta_E)}{R(\eta)} \right]^{\frac{1-3\gamma}{2}} \tag{18b}$$

$$\xi = \mu + l(1-\gamma) W_1 \int_{\eta_E}^{\eta} \left[\frac{R(\eta_E)}{R(\eta')} \right]^{\frac{1-3\gamma}{2}} d\eta' \tag{18c}$$

We are now able to evaluate the critical time for the formation of the shock. We consider separately the two cases: $\gamma = 1/3$; $\gamma \neq 1/3$. In the case $\gamma = 1$ no shock is produced.

α) $\gamma = \dfrac{1}{3}$.

We assume for the initial wave's profile on Σ

$$W_1 = -B \sin \mu \quad , \quad B > 0 \quad , \quad B = \text{const.} \tag{19}$$

Then (18c) yields:

$$\xi = \mu - e \sin \mu \tag{20}$$

where

$$e = \frac{2}{3} B l (\eta - \eta_E) > 0 .$$

Eq. (20) can be solved for μ if

$$\frac{\partial \xi}{\partial \mu} = 1 - e \cos \mu \neq 0$$

which can be violated when $e = 1$. The critical time η_c is then defined by $e = 1$. One has

$$\eta_c - \eta_E = \frac{3}{2 l B} .$$

Also, we can write:

$$\frac{\underset{(1)}{\rho}}{\underset{(0)}{\rho}} = \left(\frac{\delta\rho}{\rho}\right) \sin \mu = \frac{1+\gamma}{\gamma} \frac{l}{l_0} B \sin \mu$$

where $\dfrac{\delta p}{\rho}$ denotes the amplitude of the density contrast. It follows:

$$\eta_c - \eta_E = \frac{6}{\sqrt{3}} \frac{1}{l\left(\dfrac{\delta\rho}{\rho}\right)_E} . \tag{21}$$

Therefore, as it was to be expected, η_c decreases as the frequency and the initial amplitude of the perturbations increase.

β) $\gamma \neq \dfrac{1}{3}$.

In this case $R = L\eta^{\frac{2}{3\gamma+1}} L =$ constant and we assume the same initial conditions as in α). It follows

$$\xi = \mu - e \sin \mu$$

with

$$e = \frac{(1-\gamma)(1-3\gamma)}{(1+3\gamma)} B l \eta_E^{\frac{1-3\gamma}{1+3\gamma}} \left[\eta^{\frac{2}{1-3\gamma}} - \eta_E^{\frac{2}{1-3\gamma}}\right] \tag{22}$$

whence, from the condition $e = 1$, one finds

$$\eta_c^{\frac{2}{1-3\gamma}} = \eta_E^{\frac{2}{1-3\gamma}} + \frac{(1+3\gamma)\,\eta_E^{\frac{1-3\gamma}{1+3\gamma}}}{\gamma(1-\gamma)(1-3\gamma)\sqrt{\gamma}} \frac{1}{l} \frac{1}{\left(\frac{\delta\rho}{\rho}\right)_E}$$

which shows the same qualitative behaviour as (21). We remark that for $\gamma \to 0$, $\eta_c \to \infty$.

Now we are able to discuss the distorsion of the signal. This arises because we express μ as function of ξ, by inverting eq. (20). The latter is the well-known Kepler equation of Celestial Mechanics [10], and its solution is, for $e < 1$ (i.e. $\eta < \eta_c$):

$$\mu = \xi + \sum_{q=1}^{\infty} \frac{2 J_q(qe)}{q} \sin q\xi \tag{24}$$

where J_q is the Bessel function of order q.

Therefore we see that an initially sinusoidal profile, $W_1 = - B \sin \mu$ is subsequently distorted by the creation of the higher order harmonics. Then one has:

$$V_1 = -\frac{1}{e} B \left[\frac{R(\eta_E)}{R(\eta)}\right]^{\frac{1-3\gamma}{2}} \sum_{q=1}^{\infty} \frac{2 J_q(qe)}{q} \sin(q\xi). \tag{25}$$

As an example we treat the case $\gamma = 1/3$. Then we obtain for the density perturbations

$$\frac{\underset{(1)}{\rho}}{\underset{(0)}{\rho}} = 12\sqrt{3} \frac{1}{l(\eta - \eta_E)} \sum_{q=1}^{\infty} \frac{J_q(qe)}{q} \sin(q\xi) \tag{26}$$

which shows how the amplitude is transferred from the low to the high-frequencies.

For the short time intervals, $\eta - \eta_E \ll \eta_c - \eta_E$, one has $e \ll 1$, hence:

$$J_q(qe) \sim \frac{q^q e^q}{2^q q!}.$$

It follows that the amplitude of the mode with $q = 1$ is approximately constant, whereas that of the higher modes, $q > 1$, increases as $(\eta - \eta_E)^{q-1}$.

SECTION 3

In the particular case $\gamma = 1/3$, Liang [6] and Jones [11] found exact solutions to the hydrodynamical equations in the form of simple waves. These are one-dimensional solutions such that the Riemann invariants are constant throughout space-time [12].

In our formalism these solutions read, for propagation along the x-axis:

$$v = f(\tilde{\mu}\pm) \tag{27a}$$

$$\rho = \underset{(0)}{\rho}\left(\frac{1+v}{1-v}\right)^{\pm\frac{2}{\sqrt{3}}} \tag{27b}$$

$$\tilde{\mu}\pm = x - \frac{v \pm c_s}{1 \pm c_s v}(\eta - \eta_E) \tag{27c}$$

where c_s is the sound speed and v is defined by:

$$u^a = \frac{1}{R}(1, v, 0, 0). \tag{27d}$$

In order to compare Liang's results with our ones we must assume $v \ll 1$, because our method is essentially a perturbative one. For the sake of simplicity we consider outgoing waves. Then, from eqs. (27a, b, c, d) we obtain, up to terms of order $o(v^2)$

$$\frac{\underset{(1)}{\rho}}{\underset{(0)}{\rho}} = -\frac{4}{\sqrt{3}}v \tag{28}$$

which coincides with our result for $\dfrac{l_0}{l} = \dfrac{1}{\sqrt{3}}$.

Furthermore (27c) yields:

$$\tilde{\mu}_- = x + c_s(\eta - \eta_E) - \frac{2}{3}v(\eta - \eta_E) + o(v^2). \tag{29}$$

Let us now consider an initially sinusoidal profile for v,

$$v = -B\sin(l\tilde{\mu}).$$

If we write $\xi = l[x + c_s(\eta - \eta)_E]$, $\mu = l\tilde{\mu}_-$ then (29) yields:

$$\xi = \mu - \frac{2}{3}Bl\eta\sin\mu$$

which coincides with eq. (20) for $l = \frac{2}{3}lB$.

The critical time for the formation of the shock, as computed by Liang and by Jones, is:

$$\eta_c - \eta_E = \frac{\pi}{4\,\alpha l B}$$

with $\alpha = \frac{2}{3}\,\frac{1}{l + B/\sqrt{3}}$. Since $B \ll 1$, then $\alpha \sim \frac{2}{3}$, and one recovers the same order of magnitude as our result (21).

The above considerations show that Choquet's method provides a good approximation to the exact solutions. We conclude with an interesting application to Cosmology. For perturbations corresponding to small galactic masses, $l \gg 10^3$. Furthermore, since we consider the radiation era, $\eta_E \lesssim 10^{-3}$. Hence $\Delta\eta_c \equiv \eta_c - \eta_E \leq \eta_E$. We see that high-frequency perturbations decay rapidly within an expansion time, because in our cosmological model the expansion time $\eta_{exp} = \eta$. It is convenient, at each time, to define a critical fluctuations $\left(\frac{\delta\rho}{\rho}\right)^*$ such that $\Delta\eta_c = \eta_{exp}$

$$\left(\frac{\delta\rho}{\rho}\right)^* = \frac{1}{l\eta} \simeq \frac{\lambda}{\lambda_H}$$

where λ is the perturbation's comoving wavelength and λ_H is the horizon size at time η. Then, at time η, those perturbations with $\frac{\delta\rho}{\rho} > \left(\frac{\delta\rho}{\rho}\right)^*$ degenerate into shocks.

On the other hand we known that the radiation era lasts until the equipartition time η_{eq}. After this time no perturbation can degenerate into shocks because $\gamma \to 0$ in the matter era. It follows that, if a perturbation of comoving wave-length λ si such that $\left(\frac{\delta\rho}{\rho}\right)_{eq} < \left(\frac{\delta\rho}{\rho}\right)^*_{eq}$ at the equipartition time η_{eq}, then this perturbation can never degenerate into shocks. The critical fluctuation density at η_{eq} is

$$\left(\frac{\delta\rho}{\rho}\right)^*_{eq} = \frac{\lambda}{\lambda_H} = \left(\frac{M}{M_H}\right)^{1/3}_{eq}$$

where M is the mass associated with the perturbation and M_H is the horizon mass. The final result is the following upper bound for the amplitude at η_{eq} of those perturbations which do not degenerate into shocks:

$$\left(\frac{\delta\rho}{\rho}\right)_{eq} < \left(\frac{M}{M_H}\right)^{1/3}_{eq}.$$

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