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## Theoretical aspects in the moving frame method

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**Geometria differenziale.** — *Theoretical aspects in the moving frame method.* Nota di ALEXANDRU C. NEAGU, presentata (*) dal Socio E. MARTINELLI a nome del compianto Socio B. SEGRE.

RIASSUNTO. — In questa Nota, usando campi di vettori ed un linguaggio geometricamente invariante, si ottiene una descrizione teorica dell'algoritmo del riferimento mobile negli spazi dotati di G-struttura.

It is well known that the algorithm of moving frames founded by E. Cartan [2] constitutes a powerful and elegant tool for the study of submanifolds of a homogeneous space. Gh. Gheorghiev extends this algorithm for submanifolds of a manifold with a G-structure [3]; there remains however to give the geometrical aspects of this method (masked by the language of exterior differential forms).

In this paper we give, in terms of vector fields and in an invariant geometrical language, a rigorous description and a theoretical justification of this extended algorithm. The method is exposed for a regular distribution.

1. Let X and Y be the differentiable G-spaces, where G is a compact Lie group with acts proper and free on X [1]. Lef $f: X \to Y$ be an equivariant map [1]. A MRM-*configuration* is a set $(X, f, Y, G)$. If every orbit of Y is of type G/H and there exists an equivariant retraction $\lambda$ of Y to an orbit $G \cdot y_0$ (see [1]), then $(X, f, Y, G)$ is called *reductible.*

THEOREM 1. *If* $(X, f, Y, G)$ *is a reductible* MRM-*configuration, then the structure group* G *of the principal fibre bundle* $\pi: X \to X/G$ ($X/G =$ *the orbit space of* X) *is reductible to the subgroup* H.

*Proof.* Let $\lambda: Y \to G \cdot y_0$ be an equivariant retraction to the orbit $G \cdot y_0$ of $y_0 \in Y$. Then $\lambda^{-1}(y_0) = S$ is a global slice of Y [1]. We put $X_1 = (\lambda \circ f)^{-1}(y_0)$ and by an usual argument we prove that $X_1$ is a principal fibre subbundle of X with the structure group H.

COROLLARY 1. *In the conditions of Theorem 1, the restriction of f to the reduced bundle* $X_1 \subset X$ *defines a map* J *of* X/G *in the slice* $S \subset Y$.

*Proof.* Let $x \in X_1$ and $x^* = \pi(x)$. We put $J(x^*) = f(x)$ and since $f$ is constant on the orbits of the H-space $X_1$, it follows that J is well defined.

The map J is called *invariant* of the reductible MRM-configuration $(X, f, Y, G)$. If G acts transitivelly on Y then the MRM-configuration $(X, f, Y, G)$ is called *singular.*

(*) Nella seduta del 18 novembre 1977.

THEOREM 2. *If* $(X, f, Y, G)$ *is a singular* MRM-*configuration, then the invariant* J *is constant.*

*Proof.* Since G acts transitively, then the retraction $\lambda$ is the identity of Y and hence J is the restriction of $\lambda \circ f$ to $X_1$.

The process described in Theorem 1 and Corollary 1, by which to a MRM-configuration corresponds a subbundle $X_1$ and an invariant J, is called R-algorithm. If Y is a trivial G-space then we say that $(X, f, Y, G)$ is *irreductible*.

2. Let X be a differentiable manifold and $B(X, G)$ a G-structure on X. We shall denote by $\pi^1 : H(X) \to X$ the fibre bundle of all linear frames of X, by $K_m^p$ the Grassmann manifold of all $p$-dimensional subspaces of $R^m$ ($m = \dim X$) and by $K^p(X)$ the Grassmann bundle of the tangent $p$-dimensional subspaces of X with the structure group:

$$K_p = \{ \| a_j^i \| \in L_m / a_b^{i'} = 0 , \quad b = 1, 2, \cdots, p ;$$
$$i' = p + 1, p + 2, \cdots, m ; \quad i, j = 1, 2, \cdots, m \}$$

where $L_m$ is the structure group of H (X).

Let $\Delta$ be a regular $p$-dimensional distribution on X. Then $\Delta$ defines a cross section of $K^p(X)$ and an equivariant map $A : B \to K_m^p$. We obtain a MRM-configuration $(B, A, K_m^p, G)$. Since the structure group G of $B = B(X, G)$ is reductible to the maximal compact subgroup, we suppose since the beginning that G is compact. Then, the local G-slices in $K_m^p$ exist [1] and hence $(B, A, K_m^p, G)$ is locally reductible. In order to avoid the complications, we suppose that $(B, A, K_m^p, G)$ is reductible. Then we can apply the R-algorithm and thus we obtain a subbundle $B_1 \subset B$ with the structure group ${}^1G \subset G$ and an invariant $J^1$. The invariant $J^1$ is called *invariant of order* I of the distribution $\Delta$.

Let us consider the distribution $\Delta^1 = \operatorname{Ker} J_*^1$, where $J_*^1$ is the linear tangent map of $J^1$. The ${}^1G$-structure $B_1$ and the distribution $\Delta^1$ define the MRM-configuration $(B_1, A_1, K_m^{p_1}, {}^1G)$, where $p_1 = \dim \Delta^1$ and the algorithm follows in an obvious manner until we reach one of the following cases:

1) $B_{s_1}$ is a $\{e\}$-structure,

2) $(B_{s_1-1}, A_{s_1-1}, K_m^{p_{s_1-1}}, {}^{s_1-1}G)$ is singular.

In the first case, we obtain a cross section $n : X \to B$ which is called *normalisation* of $\Delta$, and a series of invariants $J^1, J^2, \cdots, J^{s_1}$. This constitues our aim and thus the algorithm is finished. In the second case, the invariant $J^{s_1}$ is constant and hence $\Delta^{s_1} = \operatorname{Ker} J_*^{s_1}$ coincides with the tangent distribution of X. The algorithm is continued as follows: Let $\overline{\mathscr{E}}^2 B_{s_1}$ be the first semiholonomic prolongation of $B_{s_1}$ and $j^1 \Delta : X \to J^1 K^p(X)$ the first prolongation of the section $\Delta$, where $J^1 K^p(X)$ is the fibre bundle of 1-jets of local cross section of $K^0(X) \cdot j^1 \Delta$ defines an equivariant map $A_{s_1} : \overline{\mathscr{E}}^2 B_{s_1} \to T_m^1 K_m^p$, where $T^1_m K^p_m$ is the fibre of $J^1 K^p(X)$. We obtain a MRM-configuration $(\overline{\mathscr{C}}^2 B_{s_1}, A_{s_1}, T^1_m K^p_m, {}^{s_1}G_1)$ where ${}^{s_1}G_1$ is the structure group of $\overline{\mathscr{C}}^2 B_{s_1}$. Using the R-algorithm we obtain a subbundle $\overline{\mathscr{C}}^2 B_{s_1+1} \subset \overline{\mathscr{C}}^2 B_{s_1}$ and an invariant $J^{s_1+1}: X \to T^1_m K^p_m$.

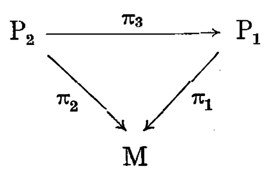
THEOREM 3. *The fibre bundle* $\overline{\mathscr{C}}^2 B_{s_1+1} \to X$ *induces a reduction of the structure group* ${}^{s_1}G$ *of* $B_{s_1}$ *to a subgroup* ${}^{s_1+1}G \subset {}^{s_1}G$.

The Theorem follows as a consequence of the following:

LEMMA. *Let* $P_\alpha(M, G_\alpha)$ $(\alpha = 1, 2)$ *be two principal fibre bundles. We suppose that*:

a) *there exists a morphism of Lie groups* $f: G_2 \to G_1$ *and a map* $\pi_3: P_2 \to P_1$ *such that* $P_2$ *is a principal fibre bundle over* $P_1$ *with the structure group* $G = \operatorname{Ker} f$;

b) *the diagram*:




*commutes*;

c) *the map* $(\pi_3, f): (P_2, G_2) \to (P_1, G_1)$ *is equivariant*.

*Then a reduction of* $G_2$ *to a closed subgroup* $H_2$ *implies a reduction of* $G_1$ *to* $H_1 = f(H_2) \subset G_2$.

*Proof*. Since $G_2$ is reductible to $H_2$, then there exists a $G_2$-equivariant map $B: P_2 \to G_2/H_2$ (see [4]) and the reduced fibre bundle is $P_3 = B^{-1}(e \cdot H_2)$. Let $C: P_1 \to G_1/H_1 \approx G_2/f^{-1}(H_1)$ be the map defined by $C(g_2 \cdot H_2) = g_2 \cdot f^{-1}(H_1) = f(g_2) \cdot H_1$. It is easy to prove that C is equivariant and, by Lemma 1 of [4], the group $G_1$ si reductible to $H_1$.

Now it is obvious which are the steps of the algorithm. For the reduction of the structural group the existence of the slices in a G-space is necessary. In this sense we prove:

THEOREM 4. *Let* $(X, f, Y, G)$ *be a* MRM-*configuration where* G *and* Y *are compacts. Then, there exists a covering* $X_i$ *of* X *such that* $(X_i, f_i, Y_i, G)$ *be a reductible* MRM-*configuration, where* $Y_i = f(X_i)$ *and* $f_i$ *be the restriction of* $f$ *to* $X_i$.

*Proof*. Since Y and G are compacts, the orbital structure of Y is finite [1]. Let this one be $\Sigma = \{H_i / i = 1, 2, \cdots, n\}$. We can suppose that $H_1 \subset H_2 \subset \cdots \subset H_n$ and that $Y_{(H_i)}$ is the submanifold of Y of all orbits of the type $G/H_i$. Then $Y_{(H_i)}$, $i = 1, 2, \cdots, n$ is a partition of Y by smooth submanifolds. We put $X_i = f^{-1}(Y_{(H_i)})$ and evidently this is a partition of X and then $(X_i, f_i, Y_{(H_i)}, G)$ are the MRM-configurations with the property that all orbits of $Y_{(H_i)}$ have the same type.

Since in the neighbourhood of every orbit of a G-space with G compact there are G-tubes (see [1]) it results that $Y_{(H_i)}$ have a covering $\{V_i^\alpha / \alpha \in I\}$ with G-tubes. Let $\{U_i^\alpha = f_i^{-1}(V_i) / \alpha \in I\}$ be the covering of $X_i$. Then $(U_i^\alpha, f_i, V_i^\alpha, G)$ are reductible MRM-configurations since in $V_i^\alpha$ there are global slices.

*Remark 1.* In the conditions of Theorem 4 we can find a finite covering with G-tubes of Y (see [1]).

*Remark 2.* If $\Delta$ is involutive then the algorithm of moving frames is applied on every integral manifold of $\Delta$ by means of the restriction of B (X , G).

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