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Oscillations caused by retarded perturbations of first order linear ordinary differential equations

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Equazioni differenziali ordinarie. — *Oscillations caused by retarded perturbations of first order linear ordinary differential equations.* Nota di G. S. LADDE, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si danno teoremi sul comportamento oscillatorio delle soluzioni di una classe di equazioni differenziali ordinarie del primo ordine con argomento ritardato.

1. INTRODUCTION

Recently, the study of oscillatory behavior of first order delay differential equations has been a subject of great interest for many investigations [1–4]. A characteristic feature of the work is that when delay vanishes the resulting first order ordinary differential equation has no oscillatory behavior. In the literature [1–4], the work in this direction is centered around the functional differential equations in which the rate functions depend on only past-history of a state of a system, however, in general, the rate functions not only depend on past-history of a state of a system but also its present state.

In the present work, we give sufficient conditions for oscillatory behavior of first order functional differential equations in which the rate functions depend on both past-states as well as the present state of a system. Our results are based on very recent and interesting results of Ladas [2]. Furthermore, our results extend and generalize the results of Ladas [2], Ladas, Lakshmikantham and Papadakis [3] and Tomaras [5] in a unified way.

2. MAIN RESULTS

Consider the functional differential equation with retarded argument

$$y'(t) + p(t) y(t) + \sum_{i=1}^{n} p_i(t) y(g_i(t)) = 0 , \tag{1}$$

where,

$$p , p_i , g_i \in C[R_+ , R] , p_i(t) \geq 0 , i \in I = \{1 , 2 , \cdots, n\} ;$$

$$g_i(t) < t, \text{ increasing in } t, \text{ and } \lim_{t\to\infty} g_i(t) = \infty \text{ for } i \in I; \; R_+ = [0 , \infty). \tag{2}$$

(*) Nella seduta del 18 novembre 1977.

A now-constant solution $y(t)$ of (1) is said to be oscillatory if it has arbitrarily large zeros on R_+. Otherwise, $y(t)$ is said to be non-oscillatory.

First, we shall give a sufficient condition for oscillatory behavior of

$$z'(t) + \sum_{i=1}^{n} q_i(t)\, z(g_i(t)) = 0 \,, \tag{3}$$

where g_i are as defined in (1); $q_i \in C[R_+, R_+]$.

THEOREM 1. *Assume that*

$$\liminf_{t\to\infty} \left[\sum_{i\in I} \int_{g^*(t)}^{t} q_i(s)\, ds \right] > \frac{1}{e} \,, \tag{4}$$

where $g^*(t) = \max_{i\in I} g_i(t)$. *Then every solution of* (3) *oscillates.*

Proof. Let $z(t)$ be a nonoscillatory solution of (3). Then, for sufficiently large $t_0 > 0$ and without loss in generality, $z(t) > 0$ for $t \geq t_0$. Because of (2), there exists $t_1 \geq t_0$ such that $z(g_i(t)) > 0$ for $t \geq t_1$ and $i \in I$. In view of (3) and the nature of q_i, we have $z'(t) < 0$ for $t \geq t_1$. From (2), we can find $t_2 \geq t_1$ such that $g_i(t_2) \geq t_1$ for $i \in I$. Hence $z(t) < z(g_i(t))$ for $t \geq t_2$ and $i \in I$. Set

$$w(t) = \frac{z(g^*(t))}{z(t)} \qquad \text{for} \quad t \geq t_2 \,. \tag{5}$$

Note that $w(t) > 1$. Dividing both sides of (3) by $z(t)$, and noting the fact that $z(t) < z(g^*(t)) < z(g_i(t))$ for $t \geq t_2$, we obtain

$$\frac{z'(t)}{z(t)} + \sum_{i\in I} q_i(t)\, w(t) < 0 \qquad \text{for} \quad t \geq t_2 \,. \tag{6}$$

Again, from (2), we find $t_3 \geq t_2$ such that $g_i(t_3) \geq t_2$ for $i \in I$. Integrating both sides of (6) from $g^*(t)$ to t, $t \geq t_3$, we get

$$\log z(t) - \log z(g^*(t)) + \sum_{i\in I} \int_{g^*(t)}^{t} q_i(s)\, w(s)\, ds \leq 0 \,,$$

and hence

$$\log w(t) \geq \sum_{i\in I} \int_{g^*(t)}^{t} q_i(s)\, w(s)\, ds \,, \qquad t \geq t_3 \,. \tag{7}$$

Set

$$\liminf_{t\to\infty} w(t) = L \,. \tag{8}$$

Since $w(t)$ is bounded below by 1, therefore $L \geq 1$. Hence, L is either finite of infinite. In the following, we show that none of these cases are valid. Thus establishing the proof of the theorem.

Case 1. Assume that L is finite. Then there would exist a sequence $\{t_n\}$, $t_n \geq t_3$ such that $t_n \to \infty$ and $w(t_n) \to \mathrm{L}$ as $n \to \infty$. From (7) and integral mean-value theorem, we have

$$\log w(t_n) \geq \sum_{i \in \mathrm{I}} \int_{g^*(t_n)}^{t_n} q_i(s)\, w(s)\, \mathrm{d}s \tag{9}$$

$$\geq w(\eta_n) \sum_{i \in \mathrm{I}} \int_{g^*(t_n)}^{t_n} q_i(s)\, \mathrm{d}s\,,$$

where $g^*(t_n) < \eta_n < t_n$, $n = 1, 2, \cdots$, and $\lim_{t \to \infty} w(\eta_n) = \mathrm{L}_1$. Noting the fact that $\mathrm{L}_1 \geq \mathrm{L}$ and taking limi both sides of (9), we obtain

$$\log \mathrm{L} \geq \mathrm{L} \lim_{n \to \infty} \left[\sum_{i \in \mathrm{I}} \int_{g^*(t_n)}^{t_n} q_i(s)\, \mathrm{d}s \right]$$

which reduces to

$$\frac{\log \mathrm{L}}{\mathrm{L}} \geq \liminf_{t \to \infty} \left[\sum_{i \in \mathrm{I}} \int_{g^*(t)}^{t} q_i(s)\, \mathrm{d}s \right]. \tag{10}$$

From the fact that $\max_{\mathrm{L} \geq 1} \dfrac{\log \mathrm{L}}{\mathrm{L}} = \dfrac{1}{e}$, the relation (10) implies

$$\frac{1}{e} \geq \liminf_{t \to \infty} \left[\sum_{i \in \mathrm{I}} \int_{g^*(t)}^{t} q_i(s)\, \mathrm{d}s \right]$$

which contradicts the hypothesis. Therefore, L is not finite.

Case 2. Assume that L is infinite. From (5) and (8), we have

$$\lim_{t \to \infty} \left[\frac{z(g^*(t))}{z(t)} \right] = \infty\,. \tag{11}$$

Choose $t_* = h(t)$ such that $g^*(t) < t_* < t$ for $t \geq t_4$, where $t_4 \geq t_3$ and $g_i(t_4) \geq t_3$ for $i \in \mathrm{I}$. From (2), we note that $g^*(t_*) < g^*(t)$. Now, integrating both sides of (3) from t_* to t and $g^*(t)$ to t_*, we obtain

$$z(t) - z(t_*) + \sum_{i \in \mathrm{I}} \int_{t_*}^{t} q_i(s)\, z(g_i(s))\, \mathrm{d}s = 0\,, \qquad t \geq t_4 \tag{12}$$

and

$$z(t_*) - z(g^*(t)) + \sum_{i \in \mathrm{I}} \int_{g^*(t)}^{t} q_i(s)\, z(g_i(s))\, \mathrm{d}s = 0\,, \qquad t \geq t_4\,, \tag{13}$$

respectively. We observe that for $s \in [t_*, t]$, $g_i(s) < g_i(t) < g^*(t)$ and $z(g^*(t)) < z(g_i(t)) < z(g_i(s))$, and for $s \in [g^*(t), t_*]$, $g_i(s) < g_i(t_*) < g^*(t_*)$

and $z(g^*(t_*)) < z(g_i(t_*)) < z(g_i(s))$. Hence, the equations (12) and (13) become

$$z(t) + z(g^*(t)) \sum_{i \in I} \int_{t_*}^{t} q_i(s)\, ds \leq z(t_*), \qquad t \geq t_4, \tag{14}$$

and

$$z(t_*) + z(g^*(t_*)) \sum_{i \in I} \int_{g^*(t)}^{t_*} q_i(s)\, ds \leq z(g^*(t)), \qquad t \geq t_4, \tag{15}$$

respectively. By dividing both sides of (14) and (15) by $z(t)$ and $z(t_*)$, respectively, and by using (4) and (11), we arrive at

$$\lim_{t \to \infty} \frac{z(t_*)}{z(t)} = \lim_{t_* \to \infty} \frac{z(g^*(t))}{z(t_*)} = \infty. \tag{16}$$

Dividing both sides of (14) by $z(t_*)$, we obtain

$$\frac{z(t)}{z(t_*)} + \frac{z(g^*(t))}{z(t_*)} \sum_{i \in I} \int_{t_*}^{t} q_i(s)\, ds \leq 1, \qquad t \geq t_4. \tag{17}$$

Because of (4) and (16), the relation (17) is impossible. Therefore, L is not infinite. Thus, the impossibility of both the cases establishes the proof of the theorem.

Remark 1. From (3), several particular cases can be derived by choosing a triplet $(n, q_i(t), g_i(t))$ such as:

(18) $(n, q_i(t), t - \tau_i)$, where, $\tau_i > 0$, and q_i and are as in (3);

(19) $(n, q_i, t - \tau_i)$, where, $q_i \geq 0$ and are constants, $\tau_i > 0$, and for some $k \in I$, $q_k > 0$;

(20) $(1, q_1(t), g_1(t))$, where, q_1 and g_1 are as in (3);

(21) $(1, q_1(t), t - \tau)$, where, $\tau > 0$ and q_1 as in (3);

(22) $(1, q, t - \tau)$, where, $\tau > 0$ and $q > 0$ is a positive constant.

Corresponding to the triplets (18), (19), (20), (21) and (22), the differential equation with retarded argument (3) reduces to

$$z'(t) + \sum_{i=1}^{n} q_i(t)\, z(t - \tau_i) = 0, \tag{23}$$

$$z'(t) + \sum_{i=1}^{n} q_i\, z(t - \tau_i) = 0, \tag{24}$$

$$z'(t) + q_1(t)\, z(g_1(t)) = 0, \tag{25}$$

$$z'(t) + q_1(t)\, z(t - \tau) = 0, \tag{26}$$

and

$$z'(t) + qz(t - \tau) = 0\,, \tag{27}$$

respectively. From (4), the sufficient condition for oscillatory behavior of (23), (24), (25), (26) and (27) is

$$\liminf_{t\to\infty} \left[\sum_{i\in I} \int_{t-\tau}^{t} q_i(s)\, ds\right] > \frac{1}{e}\,, \qquad \text{where} \quad \tau = \min_{i\in I}\{\tau_i\}\,, \tag{28}$$

$$\tau e \sum_{i\in I} q_i > 1\,, \qquad \text{where } \tau \text{ is defined in (28)}\,, \tag{29}$$

$$\liminf_{t\to\infty} \left[\int_{g_1(t)}^{t} q_1(s)\, ds\right] > \frac{1}{e}\,, \tag{30}$$

$$\liminf_{t\to\infty} \left[\int_{t-\tau}^{t} q_1(s)\, ds\right] > \frac{1}{e}\,, \tag{31}$$

and

$$\tau e q > 1\,, \tag{32}$$

respectively. From above analysis, one can easily see that Theorem 1 generalizes Theorem 1 in [2] in a natural way. Further we note that the proof of Theorem 1 is very much close to the proof of Theorem 1 in [2] with certain desired modifications.

In the following, we shall present a sufficient condition for oscillatory behavior of (1).

THEOREM 2. *Assume that*

$$\liminf_{t\to\infty} \left[\sum_{i\in I} \int_{g^*(t)}^{t} p_i(s) \exp\left[\int_{g_i(s)}^{s} p(u)\, du\right] ds\right] > \frac{1}{e} \tag{33}$$

where $g^*(t) = \max_{i\in I}\{g_i(t)\}$. *Then every solution of* (1) *oscillates.*

Proof. The proof of the theorem is given by the application of Theorem 1. In order to apply Theorem 1, we treat (1) as a retarded perturbation of corresponding first order ordinary differential equation

$$x' + p(t)\, x = 0\,, \tag{34}$$

and use the elementary method of variation of parameters to reduce (1) into the form of (3).

It is obvious that a general solution of (34) is given by

$$x(t) = \exp\left[-\int_{a_0}^{t} p(u)\, du\right] c\,, \qquad t \geq a_0 \tag{35}$$

for some $a_0 \in R_+$, where c is an arbitrary constant. Let

$$y(t) = \exp\left[-\int_{a_0}^{t} p(u)\,\mathrm{d}u\right] z(t), \qquad t \geq a_0 \tag{36}$$

be a solution of (1), where $z(t)$ is an unknown function. We note that

$$y(g_i(t)) = \exp\left[-\int_{a_1}^{g_i(t)} p(u)\,\mathrm{d}u\right] z(g_i(t)) \tag{37}$$

is defined for $t \geq a_1$, where a_1 is defined by $g_i(a_1) \geq a_0$ for $i \in I$. By differentiating both sides of (36) with respect to t, for $t \geq a_1$, and using the argument in the variation of parameters technique, we have

$$z'(t) + \exp\left[\int_{a_1}^{t} p(u)\,\mathrm{d}u\right] \sum_{i \in I} p_i(t)\, y(g_i(t)) = 0, \qquad t \geq a_1 .$$

This together with (37) yields

$$z'(t) + \sum_{i \in I} p_i(t) \exp\left[\int_{g_i(t)}^{t} p(u)\,\mathrm{d}u\right] z(g_i(t)) = 0 \tag{38}$$

for $t \geq a_1$. Setting

$$q_i(t) = p_i(t) \exp\left[\int_{g_i(t)}^{t} p(u)\,\mathrm{d}u\right], \qquad i \in I . \tag{39}$$

From (39), the relations (33) and (38) reduce to

$$\liminf_{t\to\infty} \left[\sum_{i \in I} \int_{g^*(t)}^{t} q_i(s)\,\mathrm{d}s\right] > \frac{1}{e}$$

and

$$z'(t) + \sum_{i=1}^{n} q_i(t)\, z(g_i(t)) = 0,$$

respectively. Now, by applying Theorem 1, we conclude that every solution $z(t)$ of (38) is oscillatory. This together with (36) proves that every solution $y(t)$ of (1) is oscillatory. Hence the theorem is proved.

A remark similar to Remark 1 is also of much interest.

Remark 2. Depending on the form of a quartplet $(n, p_i(t), g_i(t), p(t))$, one can generate different types of functional differential equations with retarded arguments. For example, corresponding to quartplets

$$(n, p_i(t), t - \tau_i, p(t)), (n, p_i, t - \tau_i, p),$$
$$(1, p_1(t), g_1(t), p(t)), (1, p_1(t), t - \tau, p(t))$$

and $(1, p_1, t-\tau, p)$, the equation (1) takes the following form:

$$y'(t) + p(t)y(t) + \sum_{i=1}^{n} p_i(t)\,y(t-\tau_i) = 0\,, \tag{40}$$

$$y'(t) + py(t) + \sum_{i=1}^{n} p_i\,y(t-\tau_i) = 0\,, \tag{41}$$

$$y'(t) + p(t)y(t) + p_1(t)\,y(g_1(t)) = 0\,, \tag{42}$$

$$y'(t) + p(t)y(t) + p_1(t)\,y(t-\tau) = 0\,, \tag{43}$$

and

$$y'(t) + py(t) + p_1\,y(t-\tau) = 0\,, \tag{44}$$

respectively, where $p_i(t), g_i(t), p_1(t)$ and $g_1(t)$ are functions as in (1) for $i \in \mathrm{I}$; for $i \in \mathrm{I}$, p_i are non-negative constants and for some $k \in \mathrm{I}$, $p_k > 0$; $\tau_i > 0$ and are positive constants; $\tau > 0$ and $p_1 > 0$ are positive constants; p is any constant.

From (33), the sufficient condition for oscillatory behavior of (40), (41), (42), (43) and (44) is

$$\liminf_{t\to\infty} \left[\sum_{i\in\mathrm{I}} \int_{t-\tau}^{t} p_i(s) \exp\left[\int_{s-\tau_i}^{s} p(u)\,du\right] ds\right] > \frac{1}{e}, \quad \text{where } \tau = \min_{i\in\mathrm{I}} \{\tau_i\}\,, \tag{45}$$

$$\tau e \sum_{i\in\mathrm{I}} p_i \exp[\tau_i p] > 1\,, \qquad \text{where } \tau \text{ is as in (45)}\,, \tag{46}$$

$$\liminf_{t\to\infty} \left[\int_{g_1(t)}^{t} p_1(s) \exp\left[\int_{g_1(s)}^{s} p(u)\,du\right] ds\right] > \frac{1}{e}\,, \tag{47}$$

$$\liminf_{t\to\infty} \left[\int_{t-\tau}^{t} p_1(s) \exp\left[\int_{s-\tau}^{s} p(u)\,du\right] ds\right] > \frac{1}{e}\,, \tag{48}$$

and

$$\tau e p_1 \exp[\tau p] > 1\,, \tag{49}$$

respectively.

From the above analysis, one can easily see that Theorem 2 covers variety of functional differential equations with retarded arguments.

From (38) and Theorem 2 in [2], it is obvious that the sufficient oscillatory condition (49) of (44) is also necessary.

Remark 3. Recall that the validity of Theorem 2 was established by treating (1) as a retarded or hereditary perturbation of (34). Similarly, one can treat (3) as a retarded perturbation of

$$x'(t) = 0\,. \tag{50}$$

Note that every solution $x(t) = \exp\left[-\int\limits_{a_0}^{t} p(u)\,du\right] c$ and $x(t) = c$ of unperturbed ordinary differential equations (34) and (50) is non-oscillatory, respectively. In the light of this, Theorems 1, 2 can be considered as the oscillatory results generated by constantly acting retarded or hereditary perturbations.

On the other hand, note that the equation (1) can be considered as ordinary perturbation of corresponding first order functional differential equation with retarded argument

$$x'(t) + \sum_{i=1}^{n} p_i(t)\, x(g_i(t)) = 0 . \tag{51}$$

We assume that

$$\liminf_{t\to\infty} \left[\sum_{i\in I} \int\limits_{g^*(t)}^{t} p_i(s)\,ds\right] > \frac{1}{e} . \tag{52}$$

This together with the application of Theorem 1, one can conclude that every solution of (51) is oscillatory. It is obvious that (52) implies (33), whenever

$$\liminf_{t\to\infty} \left[\int\limits_{g^*(t)}^{t} p(s)\,ds\right] \geq 0 . \tag{53}$$

Therefore, Theorem 2 preserves the oscillatory behavior of unperturbed retarded equation (51) under constantly acting ordinary perturbations $p(t)\,y(t)$. This means that ordinary perturbations $p(t)\,y(t)$ of (51) are harmless with regard to oscillatory behavior of (51), provided $p(t)$ is a continuous function on R_+, and satisfies the relation (53).

These very elementary but important observations may open up a new avenue in the oscillation theory.

Remark 4. Note that the oscillatory behavior of (1) or (3) are caused by the retarded argument. This oscillatory behavior disappears whenever retarded effects disappear.

Remark 5. From the proof of Theorem 2, one can easily see that the method of variation of parameters has played an important role in order to establish the oscillatory behavior of (1). It seems to me that the method of variation of parameters is a promising tool to study the oscillatory behavior of differential equations. However, the full force of the method remains to be seen in the future.

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