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Controllability of Perturbed Nonlinear System

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Equazioni differenziali ordinarie. — *Controllability of Perturbed Nonlinear System.* Nota(*) di JERALD P. DAUER, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si trova una condizione sufficiente per la controllabilità forte di sistemi non lineari di controlli.

1. INTRODUCTION

The controllability of various nonlinear control systems has been studied by a number of authors. One type of method used in many of these studies has been perturbation techniques (for references see [1]). In particular, the results of Dauer [1] give sufficient conditions for controllability of perturbed quasi-linear systems.

In this paper we use a nonlinear perturbation approach to obtain sufficient conditions for controllability of the more general nonlinear system

$$\dot{x} = g(t, x) + B(t, x) u + f(t, x, u) . \tag{1}$$

Our procedure first characterizes appropriate solutions of (1) using Alekseev's variation of parameters formula [3]. This method was also used by Lukes [2] to obtain results for bounded perturbations $f(t, x, u)$ of the base system

$$\dot{x} = g(t, x) + B(t, x, u) u . \tag{2}$$

He used a fixed point argument for an appropriate nonlinear operator defined on a Banach space. Our approach is similar, although our operator differs in an essential manner from that used by Lukes allowing a more general class of perturbations.

The sufficient conditions for controllability which we develop for system (1) are for a class of nonlinear perturbations $f(t, x, u)$ which satisfy a "less than linear growth" condition, a condition which all bounded functions satisfy. The motivation for a condition of this type on $f(t, x, u)$ can be seen from the discussion and linear examples of Lukes [4]. The conditions on the base system (2) are that it satisfy a strong controllability condition and that $|\partial g/\partial x|$ and $|B|$ are bounded. Completely controllable linear systems are examples of such base systems.

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2. Controllability results

Consider the nonlinear control system (1) for $t \in I = [t_0, t_1]$ where $x \in E^n$, Euclidean n-space, $u \in E^m$ and g, B and f are continuous functions of appropriate dimensions. We say that system (1) is *completely controllable* if for any $x_0, x_1 \in E^n$ there exists a continuous control function $u(t)$ such that the solution of

$$\begin{aligned} \dot{x} &= g(t, x) + B(t, x) u(t) + f(t, x, u(t)) \\ x(t_0) &= x_0 \end{aligned} \tag{3}$$

satisfies $x(t_1) = x_1$.

In order to obtain a usable form for the solution of (3) we assume that g, B and f satisfy the following basic continuity and boundedness condition:

(C 1) Let $g(t, x)$ be twice continuously differentiable in x and once in t, $B(t, x)$ be once continuously differentiable in x and $|\partial g/\partial x|$ be bounded on $I \times E^n$.

Then there exists a unique solution $y(t, s, x_0)$ of

$$\dot{y} = g(t, y)$$
$$y(s, s, x_0) = x_0$$

defined on I [2, 3]. It follows that the corresponding Jacobi matrix function

$$Z(t, s, x) = \frac{\partial y(t, s, x)}{\partial x}$$

is bounded on $I \times I \times E^n$ and is the fundamental matrix solution of

$$\frac{\partial Z}{\partial t} = \left[\frac{\partial g(t, y(t, s, x))}{\partial y}\right] Z$$

such that $Z(t, t, x)$ is the identity matrix. By Alekseev's variation of parameters formula [2, 3], for every continuous (control) function $u(t)$ the unique solution of (3) is given by

$$\begin{aligned} x(t) = y(t, t_0, x_0) &+ \int_{t_0}^{t} Z(t, s, x(s)) B(s, x(s)) u(s) \, ds \\ &+ \int_{t_0}^{t} Z(t, s, x(s)) f(s, x(s), u(s)) \, ds . \end{aligned} \tag{4}$$

In particular, it is easy to see that a solution $x(t)$ of (3) satisfying $x(t_1) = x_1$ corresponds to the control function defined by

$$u(t) = B^*(t, x(t)) Z^*(t_1, t, x(t)) S^{-1}(x, u) p(x, u) \tag{5}$$

where

$$S(x, u) = \int_{t_0}^{t_1} \psi(t) \psi^*(t) \, dt ,$$

$$\psi(t) = Z(t_1, t, x(t)) B(t, x(t)) ,$$

$$p(x, u) = x_1 - y(t_1, t_0, x_0) - \int_{t_0}^{t_1} Z(t_1, t, x(t)) f(t, x, (t), u(t)) \, dt ,$$

here * denotes matrix transposition.

We now determine conditions on system (1) which guarantee that for every pair of points x_0, x_1 there is a pair of continuous functions $x(t), u(t)$ which satisfies the set of integral equations (4), (5). This result extends those [2] for system (1) by eliminating the boundedness hypothesis on the partial derivatives of B and by enlarging the class of perturbations. Our proof follows that used by Dauer [1] for perturbations of quasi-linear control systems.

We say that system (2) satisfies a *strong controllability condition* if there exists a number $\lambda > 0$ such that for any pair of continuous functions $x(t), u(t)$ and all $w \in E^n$ we have

$$w^* S(x, u) w \geq |w|^2 .$$

For such systems it follows that the symmetric and nonnegative matrix $S(x, u)$ has an inverse which is bounded

$$|S^{-1}(x, u)| \leq 1/\lambda$$

independently of the functions $x(t), u(t)$. For linear systems this reduces to the standard necessary and sufficient condition for complete controllability developed by Kalman, Ho and Narendra [5]. It follows from the result below that systems (2) which satisfy a strong controllability condition are completely controllable provided g and B satisfy condition (C 1) and $|B(t, x)|$ is bounded on $I \times E^n$. This extends the results of Davison and Kunze [6] for this system.

THEOREM. *Suppose that* g, B *and* f *satisfy the basic continuity and boundedness conditions* (C 1) *and that* $|B(t, x)|$ *is bounded on* $I \times E^n$. *If the base system* (2) *satisfies a strong controllability condition and the perturbation* f *is such that*

$$\lim_{|(x,u)| \to \infty} \frac{|f(t, x, u)|}{|(x, u)|} = 0$$

uniformly for $t \in I$, *then system* (1) *is completely controllable.*

Proof. Let C denote the Banach space of continuous functions $(x, u)\colon I \to E^n \times E^m$ with the usual sup norm,

$$\|(x, u)\| = \sup \{ |(x(t), u(t))| : t \in I \} .$$

Fix $x_0, x_1 \in E^n$ and define a continuous operator T on C as follows: for each $(x, u) \in C$ let $T(x, u) = (w, v)$ where

$$v(t) = B^*(t, x(t)) Z^*(t_1, t, x(t)) S^{-1}(x, u) p(x, u), \tag{6}$$

$$w(t) = y(t, t_0, x_0) + \int_{t_0}^{t} Z(t, s, x(s)) B(s, x(s)) v(s) ds \tag{7}$$

$$+ \int_{t_0}^{t} Z(t, s, x(s)) f(s, x(s), u(s)) ds .$$

Take

$$\begin{aligned} k &= \max \{\| Z \| \cdot \| B \| (t_1 - t_0), 1\}, \\ c_1 &= 4 k \| B^* \| \cdot \| Z^* \|^2 (t_1 - r_0)/\lambda, \\ d_1 &= 4 k \| B^* \| \cdot \| Z^* \| \cdot | x_1 - y(t_1, t_0, x_0) | /\lambda, \\ c_2 &= 4 \| Z \| (t_1 - t_0), \\ d_2 &= 4 | y(t_1, t_0, x_0) |, \\ c &= \max \{c_1, c_2\}, \\ d &= \max \{d_1, d_2\}. \end{aligned}$$

It follows from the growth condition on f [1, Prop. 1] that there exists a constant r such that if $|(x, u)| \leq r$ and $s \in I$ then

$$c |f(s, x, u)| + d \leq r .$$

Letting $C_r = \{(x, u) \in C : \| (x, u) \| \leq r\}$ we have that if $(x, u) \in C_r$ and $T(x, u) = (w, v)$, then

$$\begin{aligned} \| v \| &\leq [d_1 + c_1 \sup_{s \in I} | f(s, x(s), u(s)) |]/(4 k) \\ &\leq r/(4 k) \leq r/4 . \end{aligned}$$

Hence

$$\begin{aligned} \| w \| &\leq d_2/4 + k \| v \| + (c_2/4) \sup_{s \in I} | f(s, x(s), u(s)) | \\ &\leq r/4 + r/4 . \end{aligned}$$

Therefore, T maps C_r into itself. In particular, T maps the convex closure of $T[C_r]$ into itself. Let W_r be the set of all functions w which are defined by equation (7) for $(x, u) \in C_r$ with $v(s)$ defined by (6). Since f, and therefore p, is bounded on C_r and Z, B, S^{-1} and g are bounded, it follows that W_r is equicontinuous. Therefore, the range of the product function $B^* Z^*$ defined on $T[C_r]$ is equicontinuous. Hence, equations (6), (7) show that $T[C_r]$ is equicontinuous and the Schauder-Tychonoff Theorem [7] shows that T has a fixed point in C_r. This fixed point (x, u) of T is a solution pair of the set of integral equations (4), (5). Hence $u(t)$ is a control function whose corresponding solution $x(t)$ of (3) satisfies $x(t_1) = x_1$. This proves the result. □

The condition that the base system (2) satisfy a strong controllability condition is a difficult condition to verify for nonlinear systems in general. However, Davison and Kunze [6] have developed several examples of such systems. In particular, their results [6, Theorem 4] with the above theorem give the following.

COROLLARY. *Consider the system*

$$\dot{x} = A(t, x) x + B(t, x) u + f(t, x, u) \tag{8}$$

where

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ a_{n-2,1} & & & \cdots & a_{n-2,n-1} & 0 \\ a_{n-1,1} & & & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & & & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ b \end{bmatrix}$$

The system (8) *is completely controllable provided the following conditions are satisfied*:

i) *The first* $n - 2$ *partial derivatives of* $A(t, x)$ *and* $n - 1$ *derivatives of* $B(t, x)$ *exist for* $t \in I$ *and for* $x \in E^n$,

ii) $|A(t, x)|$ *and* $|B(t, x)|$ *are bounded on* $I \times E^n$,

iii) *There exists a constant* $c > 0$ *and a point* $t_j \in I$ *such that*

$$b^2(t_j, x) \geq c, \quad a^2_{i,i+1}(t_j, x) \geq c$$

for all $x \in E^n$ *and* $i = 1, 2, \cdots, n - 1$,

iv) $f(t, x, u)$ *is once continuously differentiable in* x *and satisfies*

$$\lim_{|(x,u)| \to \infty} \frac{|f(t, x, u)|}{|(x, u)|} = 0$$

uniformly for $t \in I$.

Examples of such control systems can be easily constructed from the following nth order nonlinear control system (see also [8]), here $u, y \in E^1$,

$$\begin{aligned} y^{(n)}(t) &+ a_1(t, y(t), \dot{y}(t), \cdots, y^{(n-1)}(t))\, y^{(n-1)}(t) \\ &+ \cdots + a_n(t, y(t), \dot{y}(t), \cdots, y^{(n-1)}(t))\, y(t) \\ &= b(t, y(t), \dot{y}(t), \cdots, y^{(n-1)}(t))\, u(t) \\ &+ f(t, y(t), \dot{y}(t), \cdots, y^{(n-1)}(t), u(t)) . \end{aligned}$$

Remark. A result corresponding to the above theorem is also valid for nonlinear perturbations of system (2) when the derivatives $B_t(t, x, u)$, $B_x(t, x, u)$ and $B_u(t, x, u)$ are bounded on $I \times E^n \times E^m$. This type of system was analyzed by Lukes for bounded perturbations. The proof follows that above with the operator T defined on $C_r \cap L_k$, where

$$L_k = \{z \in C : | z(t+\varepsilon) - z(t) | \leq k | \varepsilon | , \varepsilon > 0 , t \in I\} .$$

The additional details follow those of Lukes [2, p. 52].

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