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Relaxation of non convex variational problems

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Calcolo delle variazioni. — *Relaxation of non convex variational problems* (*). Nota di PAOLO MARCELLINI (**) e CARLO SBORDONE (***), presentata (****) dal Socio C. MIRANDA.

RIASSUNTO. — Si danno condizioni necessarie affinché un integrale del calcolo delle variazioni risulti sequenzialmente semicontinuo inferiormente nella topologia debole di $H^{1,\alpha}$ e si prova che il massimo funzionale semicontinuo inferiormente minorante è ancora un integrale del calcolo delle variazioni. Ne consegue un teorema di « rilassamento » nel senso di Ekeland e Temam [1].

Let $f = f(x, s, \xi)$ be a measurable function on $R^n \times R \times R^n$ such that

$$0 \leq f(x, s, \xi) \leq a(x) + b\,|s|^\alpha + c\,|\xi|^\alpha, \tag{1}$$

where $\alpha \geq 1$; $a \in L^1_{loc}$; $b, c > 0$.

For Ω bounded open set in R^n we consider the functional

$$u \in H^{1,\alpha}(\Omega) \rightarrow \int_\Omega f(x, u, \operatorname{grad} u)\, dx\,; \tag{2}$$

we have the following

THEOREM 1. *Let f satisfy* (1) *and be continuous in s. Let $u_0 \in H^{1,\alpha}(\Omega)$. If* (2) *is sequentially lower semicontinuous (s.l.s.) on $u_0 + H_0^{1,\alpha}(\Omega)$ in the weak topology of $H^{1,\alpha}(\Omega)$, then $f(x, s, \xi)$ is convex in $\xi \in R^n$ for almost all $x \in \Omega$ and for any $s \in R$.*

The preceding result is known with different assumptions (see e.g. Theorem 4.4.2 and 4.4.3 in [2]). Let us observe explicitly that in Theorem 1 we do not assume the continuity of f with respect to x and ξ and that we require (2) to be semicontinuous for an open set only, and we limit ourselves to the functions of $H^{1,\alpha}(\Omega)$ with a prescribed value on $\partial\Omega$.

If f is *not convex* in ξ, it is interesting to characterize the s.l.s. envelope of (2) in the weak topology of $H^{1,\alpha}(\Omega)$.

To this aim for any $(x, s) \in R^n \times R$ let us denote by

$$\xi \rightarrow f^{**}(x, s, \xi) \quad \textit{the greatest convex minorant of} \quad \xi \rightarrow f(x, s, \xi). \tag{3}$$

Remark. The supremum of a family of convex functions being a convex function, the existence of f^{**} follows. But, passing from f to f^{**}, what one generally looses is the continuity in s.

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Precisely, if f is a Caratheodory function (i.e. measurable in x and continuous in (s, ξ)) it is not necessary for f^{**} to be such.

Nevertheless, it is possible to check that f^{**} is a Caratheodory function in each of the following cases:

(i) f is coercive, e.g. as in the next (12),

(ii) f is independent of s,

(iii) f is continuous in s uniformly for $\xi \in R^n$.

The following theorem generalizes a result obtained with a different method in [1], in the case (ii).

THEOREM 2. *If f and f^{**} are Caratheodory mappings (see the preceding remark), f verifying* (1), *then, fixed a bounded open set $\Omega \subset R^n$ and $u_0 \in H^{1,\alpha}(\Omega)$, the greatest sequentially lower weak semicontinuous functional less than* (2) *on $H^{1,\alpha}(\Omega)$ is*

$$u \in H^{1,\alpha}(\Omega) \to \int_\Omega f^{**}(x, u, \operatorname{grad} u)\, dx. \tag{4}$$

Such a functional is also the s.l.s. envelope of (2) *in the weak topology on the whole of $H^{1,\alpha}(\Omega)$.*

Let us indicate a sketch of the proof of Theorem 2.

First we limit ourselves to the functions on R^n with a fixed Lipschitz constant. So, if $r > 0$, and $u \in L^1_{loc}$ satisfies $\| \operatorname{grad} u \|_{L^\infty(R^n)} \le r$, for any bounded open set $\Omega \subset R^n$, let us define

$$\overline{F}(r, \Omega, u) = \inf \left\{ \liminf_h \int_\Omega f(x, u_h, \operatorname{grad} u_h)\, dx : u_h \to u \text{ in } C^0(\overline{\Omega}), \| Du_h \|_{L^\infty(R^n)} \le r \right\} \tag{5}$$

$$\overline{F}_0(r, \Omega, u) = \inf \left\{ \liminf_h \int_\Omega f(x, u_h, \operatorname{grad} u_h)\, dx : (u_h - u) \to 0 \text{ in } C^0_0(\Omega), \| Du_h \|_{L^\infty(R^n)} \le r \right\}. \tag{6}$$

By compactness reasons the C^0 topology may be replaced by any topology weaker than the weak-star one on $H^{1,\infty}(\Omega)$.

In the class of functions with a Lipschitz constant not greater than r, the functional $\overline{F}(r, \Omega, \cdot)$ is the lower semicontinuous envelope of (2) with respect to the above topologies and $\overline{F}_0(r, \Omega, \cdot)$ is the similar one in the subclass of the functions with a prescribed value on $\partial\Omega$.

The idea is to study $\overline{F}$ and $\overline{F}_0$ as functions of Ω. One has that

(7) $\overline{F}(r, \cdot, u)$ *is increasing,*

(8) $\overline{F}_0(r, \cdot, u)$ *is additive and interiorly absolutely continuous.*

Then it is useful to prove a priori that $\overline{F} = \overline{F}_0$, because a set-function satisfying (7), (8) is an integral, by the Radon-Nikodym theorem. As clearly $\overline{F} \leq \overline{F}_0$, in order to have $\overline{F} = \overline{F}_0$ we prove that:

(9) *For* Ω *open cube*: $\overline{F}(r, \Omega, u) \geq \overline{F}_0(r + \varepsilon, \Omega, u) \quad \forall \varepsilon > 0$.

(10) *If* $\|\operatorname{grad} u\|_{L^\infty} < r$, *then* $\lim_{\varepsilon \to 0^+} \overline{F}_0(r + \varepsilon, \Omega, u) = \overline{F}_0(r, \Omega, u)$.

The subsequent step consists in proving that $\overline{F}_0(r, \Omega, \cdot)$ is an integral of the variational calculus as (2) is. To this aim one uses the uniform continuity of $f(x, \cdot, \cdot)$ on the bounded sets of $R \times R^n$.

For this reason we have restricted ourselves to the functions with a bounded gradient, from the beginning.

By applying a variant of Theorem 1, we obtain

$$\overline{F}_0(r, \Omega, u) = \int_\Omega f_r^{**}(x, u, \operatorname{grad} u)\, dx, \tag{11}$$

where $f_r^{**}(x, s, \xi)$ is the greatest convex minorant of $f(x, s, \xi)$ on $|\xi| \leq r$.

Passing to the limit as $r \to +\infty$ in (11) we obtain the result.

Theorem 2 can be used in the study of " relaxed " problems in the sense of [1]. To this aim, let us suppose that f verifies in $\Omega \times R \times R^n$ the coercivity condition ($0 < d \leq c$)

$$d\, |\xi|^\alpha \leq f(x, s, \xi). \tag{12}$$

Fixed $u_0 \in H^{1,\alpha}(\Omega)$ let us consider the problems

(13) *to minimize* $\int_\Omega f(x, u, \operatorname{grad} u)\, dx$ *on* $u_0 + H_0^{1,\alpha}(\Omega)$

(14) *to minimize* $\int_\Omega f^{**}(x, u, \operatorname{grad} u)\, dx$ *on* $u_0 + H_0^{1,\alpha}(\Omega)$.

THEOREM 3. *Let f be a Caratheodory function satisfying* (1), (12) *with* $\alpha > 1$. *The problem in* (14) *has solutions which are the limit points of the sequences minimizing problem* (13) *in the weak topology of* $H^{1,\alpha}(\Omega)$. *Moreover* $\inf(13) = \min(14)$.

If $u_0 \in L^\infty(\Omega)$, *then the solutions of* (14) *are the limit points of the sequence minimizing* (13) *also in the strong topology of* $L^\infty(\Omega)$.

The result concerning the weak topology of $H^{1,\alpha}(\Omega)$ is an immediate consequence of Theorem 2, and it has been proved, in a different way, by Ekeland and Temam ([1], Ch. X, Corollary 3.8.).

The result concerning the $L^\infty(\Omega)$ strong topology is obtained by utilizing a maximum principle for solutions of (14) which has been recently obtained by Scheurer [3].

References

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