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On Infinite Products of Resolvents

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Analisi funzionale. — *On Infinite Products of Resolvents.* Nota di SIMEON REICH, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostrano tre risultati sui prodotti infiniti di risolventi di operatori «m–accretive» negli spazi di Banach.

Let E be a real Banach space and let I denote the identity operator. Recall that a subset A of $E \times E$ with domain D (A) and range R (A) is said to be *m*-accretive if for all $x_i \in D(A)$, $y_i \in Ax_i$, $i = 1, 2$, and $r > 0$, $|x_1 - x_2| \leq |x_1 - x_2 + r(y_1 - y_2)|$, and $R(I + rA) = E$. The resolvent $J_r : E \to D(A)$ and the Yosida approximation $A_r : E \to R(A)$ of A are defined by $J_r = (I + rA)^{-1}$ and $A_r = (I - J_r)/r$. We denote the closure and convex hull of $D \subset E$ by cl (D) and co (D) respectively. We also define $\| D \| = \inf \{ |x| : x \in D \}$.

Let $\{r_n\}$ be a positive sequence. For x in E we define $\{x_n\} \subset E$ by $x_{n+1} = J_{r_n} x_n$, $x_1 = x$. The behavior of $\{x_n\}$ when $0 \in R(A)$ has been recently considered by Rockafellar [4], Brézis and Lions [1], and Bruck and Reich [2]. In this note we present three new results which are of interest when $0 \notin R(A)$. For the case $r_n = r$ for all n, see [2, Theorem 2.4] where the argument is different.

THEOREM 1. *Suppose that the norm of* E *is uniformly Gâteaux differentiable and that the norm of* E* *is Fréchet differentiable. If* $\sum_{i=1}^{\infty} r_i = \infty$, *then the strong* $\lim_{n \to \infty} x_{n+1} \Big/ \left(\sum_{i=1}^{n} r_i \right) = -v$, *where* v *is the point of least norm in* cl (R(A)).

Proof. Since the norm of E is uniformly Gâteaux differentiable, cl (R (A)) is convex and v exists. Let ε be positive. There are $y \in D(A)$ and $z \in Ay$ such that $|z| \leq |v| + \varepsilon$. Let $y_{n+1} = \prod_{i=1}^{n} J_{r_i} y$. We have

$$|x_{n+1} - x| \leq |x_{n+1} - y_{n+1}| + |y_{n+1} - y| + |y - x| \leq 2|x - y| + \sum_{i=1}^{n} r_i |A_{r_i} y_i|$$
$$\leq 2|x - y| + \left(\sum_{i=1}^{n} r_i \right) \| Ay \| \leq 2|x - y| + \left(\sum_{i=1}^{n} r_i \right) (|v| + \varepsilon),$$

so that

$$\limsup_{n \to \infty} |x_{n+1} - x| \Big/ \left(\sum_{i=1}^{n} r_i \right) \leq |v|.$$

(*) Nella seduta del 18 novembre 1977.

Since

$$(x - x_{n+1})\Big/\left(\sum_{i=1}^{n} r_i\right) = \left(\sum_{i=1}^{n} r_i A_{r_i} x_i\right)\Big/\left(\sum_{i=1}^{n} r_i\right)$$

belongs to co(R(A)), we also have $|x - x_{n+1}|\Big/\left(\sum_{i=1}^{n} r_i\right) \geq |v|$ for all n. Thus $|x - x_{n+1}|\Big/\left(\sum_{i=1}^{n} r_i\right) \to |v|$, and the result follows because the norm of E^* is Fréchet differentiable.

The conclusion of Theorem 1 can be strengthened if additional hypotheses are made.

THEOREM 2. *Suppose that the norm of* E *is uniformly Gâteaux differentiable and that* E *is uniformly convex. If either*

(a) $\{r_n\}$ *is bounded away from zero,*

or

(b) *the modulus of convexity of* E *satisfies* $\delta(\varepsilon) \geq C\varepsilon^p$ *for some* $p \geq 2$ *and* $C > 0$, *and* $\sum_{i=1}^{\infty} r_i^p = \infty$, *then the strong* $\lim_{n\to\infty} A_{r_n} x_n = v$.

Proof. Since $\{|A_{r_n} x_n|\}$ is decreasing, $b = \lim_{n\to\infty} |A_{r_n} x_n|$ exists. If $b > |v|$, let $z \in Ay$ satisfy $|z| \leq (b + |v|)/2$. Denoting $|x - y|$ by M, we have

$$|x_{n+1} - y_{n+1}| \leq \tfrac{1}{2}\, |x_{n+1} - y_{n+1} + x_n - y_n| \leq |x_n - y_n| - \\ - \delta(|x_n - x_{n+1} - (y_n - y_{n+1})|/M)\, M\,,$$

so that

$$\sum_{i=1}^{\infty} \delta(r_n\, |A_{r_n} x_n - A_{r_n} y_n|/M) < \infty\,.$$

Since

$$|A_{r_n} x_n - A_{r_n} y_n| \geq |A_{r_n} x_n| - |A_{r_n} y_n| \geq (b - |v|)/2\,,$$

this leads to a contradiction in both cases. Thus $b = |v|$, and the result follows because $A_{r_n} x_n \in R(A)$ and E is uniformly convex.

Remark 1. Under the assumptions of Theorem 1, the conclusion of Theorem 2 does not always hold.

THEOREM 3. *In the setting of Theorem 2,* $\lim_{n\to\infty} |x_n| = \infty$ *if and only if* $0 \notin R(A)$.

Proof. It is clear that if $0 \in R(A)$, then $\{x_n\}$ is bounded. Conversely, suppose $\{x_{n_k}\}$ is bounded. Then $v = 0$ by Theorem 1 and $A_{r_n} x_n \to 0$ by Theorem 2. Hence $|x_{n_k} - J_1 x_{n_k}| \leq \|Ax_{n_k}\| \to 0$ and the asymptotic center of $\{x_{n_k}\}$ is a zero of A.

Remark 2. These results are also valid for certain accretive operators which are not necessarily m-accretive.

Remark 3. Applying similar ideas to the sequence $\{z_n\}$ defined by $z_{n+1} = (1 - c_n) z_n + c_n T z_n$, where $T : E \to E$ is nonexpansive and $0 < c_n < 1$, we can in certain cases improve upon the results of [3].

Remark 4. In Theorem 2, (a) can be replaced by (a') $\{r_n\}$ does not converge to zero.

References

[1] H. BRÉZIS and P. L. LIONS – *Produits infinis de résolvantes*, to appear.

[2] R. E. BRUCK and S. REICH – *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, « Houston J. Math. », to appear.

[3] S. REICH (1973) – *Asymptotic behavior of contractions in Banach spaces*, « J. Math. Anal. Appl. », *44*, 57–70.

[4] R. T. ROCKAFELLAR (1976) – *Monotone operators and the proximal point algorithm*, « SIAM J. Control and Optimization », *14*, 877–898.