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An approximation theorem for semigroups of operators

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Analisi funzionale. — *An approximation theorem for semigroups of operators.* Nota di T. O. ADEWOYE (*), presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Se G è un gruppo abeliano compatto; se C (G) è lo spazio di Banach dalle funzioni continue e assolutamente integrabili; se $(T(\xi) : \xi > 0)$ un operatore limitato su C (G); allora per $\xi \to 0$ sussiste una limitazione per $\| T(\xi) f - f \|$.

1. INTRODUCTION

Let G be a compact abelian group with character group $\hat{G}$, C (G) and $L_1(G)$ respectively the usual Banach spaces of continuous and absolutely integrable complex-valued functions on G, and let X denote an arbitrary, but fixed, member of the set $\{C(G), L_1(G)\}$. Let U and V be subsets of X. A complex-valued function Φ on $\hat{G}$ is called a (U ,V)-multiplier if given $f \in U$, there exists $g \in V$ such that $\hat{g}(\sigma) = \Phi(\sigma) \hat{f}(\sigma)$ for all $\alpha \in \hat{G}$. [Here $\hat{f}$ denotes the Fourier transform of f, for each $f \in X$].

Now, let ν be a complex-valued function on $\hat{G}$ such that, for each $\xi > 0$, $e^{\xi\nu}$ is an (X , X)-multiplier. We associate with each $\xi > 0$ an operator $T(\xi)$ on X, defined by

$$[T(\xi) f]^{\hat{}}(\sigma) = e^{\xi\nu(\sigma)} \hat{f}(\sigma) \tag{1.1}$$

for all $f \in X$ and $\sigma \in \hat{G}$. We investigate, in this paper, the degree of approximation of the identity operator $T(o)$ by the operator $T(\xi)$ for small values of the parameter ξ, i.e. the order of magnitude of $\| T(\xi) f - f \|$, as a function ξ. Our result generalises to compact abelian groups a portion of Hille and Phillips' result ([1], Theorem 20.6.1], proved for the circle group. Such results concerning approximation of the identity are of interest for applications to the theory of summability and singular integrals [(2], [3]).

2. PRELIMINARIES

We state a few definitions (explained in detail in [1]) concerning semigroup of operators on a Banach space.

2.1. DEFINITION. Let X be a complex Banach space and let B (X) be the complex Banach algebra of all continuous linear operator on X. For

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$\xi > 0$, let $T(\xi)$ be an operator in $B(X)$. The collection $\mathscr{I} = \{T(\xi) : \xi > 0\}$ is said to be a semigroup of operators on X if

$$T(\xi_1 + \xi_2) = T(\xi_1)\, T(\xi_2) \tag{2.1}$$

for all $\xi_1, \xi_2 > 0$, i.e. $T(\xi_1 + \xi_2)\, x = T(\xi_1)\,[T(\xi_2)\, x]$ for all $x \in X$ and $\xi_1, \xi_2 > 0$.

As X may carry the weak, strong or uniform operator topology the continuity or measurability of the operators $T(\xi)$ is defined relative to the topology on X, for instance, $\mathscr{I}$ is said to be strongly continuous if

$$\lim_{\xi \to \xi_0} \| T(\xi)\, x - T(\xi_0)\, x \| = 0 \tag{2.2}$$

for all $x \in X$ and all $\xi_0 > 0$.

The infinitesimal operator A_0 of $\mathscr{I}$ is defined by

$$A_0 x = \lim_{\xi \to 0+} \frac{1}{\xi} [T(\xi)\, x - x] \tag{2.3}$$

for all $x \in X$ for which this limit exists. The operator A_0 is in general not bounded; however, $D(A_0)$, the domain of A_0 is dense in $X_0 = \{T(\xi)\, x : x \in X, \xi > 0\}$. Moreover, A_0 is in general not closed, its closure A, when it exists, is called the infinitesimal generator of $\mathscr{I}$.

2.2. DEFINITION. Let $\mathscr{I} = \{T\{\xi\} : \xi > 0\}$ be a strongly continuous semigroup of bounded linear operators on X. $\mathscr{I}$ is said to be of class $(1, C_1)$, ([1], p. 322), if

$$\text{(i)} \qquad \int_0^1 \| T(\xi) \| \, d\xi < \infty$$

and

$$\text{(ii)} \qquad \lim_{\eta \to 0+} \frac{1}{\eta} \int_0^{\eta} T(\xi)\, x \, d\xi = x$$

in norm, for each $x \in X$.

For the basic classes of semigroups of operators on a Banach space, see 10.6 of [1].

3. Now, let X be an arbitrary, but fixed, member of $\{C(G), L_1(G)\}$ as in the introduction.

3.1. DEFINITION. Let J be a subset of $\hat{G}$. The linear extension of J, denoted by $\mathscr{L}_J$, is the set of all finite linear combinations of elements of J. $\overline{\mathscr{L}}_J$, the closure of $\mathscr{L}_J$ in the norm of X, is called the closed linear extension of J.

Since the linear extension of J is the smallest subspace of X containing all the characters $\chi_\sigma, \sigma \in J$, we see that $\overline{\mathscr{L}}_J$ is identifiable with the set of trigonometric polynomials on J ([4], (27, 8)). Moreover, if $f \in X$ is such that $\hat{f}(\sigma) = 0$ for all $\sigma \notin J$, then ([4], p. 98) there exists a sequence $\{t_n\}$ in $\overline{\mathscr{L}}_J$, such that $\| f - t_n \| \to 0$. Since $\overline{\mathscr{L}}_J$ is closed this will mean that $f \in \overline{\mathscr{L}}_J$.

We are now in a position to state our approximation theorem which generalises to compact abelian groups a portion of the result by Hille and Phillips ([1], Theorem 20.6.1), proved for the circle group.

3.2. THEOREM. *Let* G *be a compact abelian group and let* X *be an arbitrary, but fixed, member of the set* $\{C(G), L_1(G)\}$. *Suppose the operator* $T(\xi), \xi > 0$, *on* X, *defined by* (1.1) *satisfies*

$$\text{(i)} \qquad \int_0^1 \| T(\xi) \| \, d\xi < \infty$$

and

$$\text{(ii)} \qquad \lim_{\eta \to 0+} \frac{1}{\eta} \int_0^\eta T(\xi) f \, d\xi = f, \qquad \textit{for each } f \in X.$$

Then, (*a*) *Let* $J = \{\sigma \in \hat{G} : \nu(\sigma) = 0\}$. *We have*

$$\lim_{\xi \to 0+} \inf \frac{1}{\xi} \| T(\xi) f - f \| = 0 \tag{3.1}$$

iff f *belongs to the closed linear extension of* J,

(b) *Let* A *be the infinitesimal generator of* $\{T(\xi) : \xi > 0\}$. *For each* $f \in D(A)$, *the domain of* A *we have*

$$T(\xi) f - f = \xi (Af + o(1)) \tag{3.2}$$

for all $\xi > 0$;

(*c*) $D(A) = \{f \in X : \nu\hat{f} = \hat{g}$ *for some* $g \in X\}$, *i.e.* ν *is a* $(D(A), X)$-*multiplier.*

Proof (*a*). By Theorem 1.2 of [5], $\{T(\xi) : \xi > 0\}$ is a strongly continuous semigroup of operators on X. The assumptions (i) and (ii) of the theorem imply that $\mathscr{I}$ is of class $(1, C_1)$. Suppose $f \in X$ satisfies (3.1), by Theorem 10.7.2 of [1], $T(\xi) f = f$ for all $\xi > 0$. Conversely if $f \in X$ is such that $T(\xi) f = f$ for all $\xi > 0$, then it is clear that f satisfies (3.1). Thus f satisfies $\lim_{\xi \to 0+} \inf \frac{1}{\xi} \| T(\xi) f - f \| = 0$ iff $T(\xi) f = f$ for all $\xi > 0$.

Next, we show that $T(\xi) f = f$ for all $\xi > 0$ iff for each $\sigma \in \hat{G}$, $e^{\xi\nu(\sigma)} \hat{f}(\sigma)$ is independent of ξ. Suppose $T(\xi) f = f$ for all $\xi > 0$. Then for each $\sigma \in \hat{G}$, we have $[T(\xi) f]^\wedge(\sigma) = \hat{f}(\sigma)$. Since $\hat{f}(\sigma)$ is independent of ξ, it follows that

$e^{\xi\nu(\sigma)}\hat{f}(\sigma)$ is independent of ξ also. Conversely, suppose that for each $\sigma \in \hat{G}$, $e^{\xi\nu(\sigma)}\hat{f}(\sigma)$ is independent of ξ. Then $e^{\xi\nu(\sigma)}\hat{f}(\sigma) = e^{2\xi\nu(\sigma)}\hat{f}(\sigma)$, implying that $\hat{f}(\sigma) = e^{\xi\nu(\sigma)}\hat{f}(\sigma)$ for each $\sigma \in G$. We then have $[T(\xi)f]^{\wedge}(\sigma) = \hat{f}(\sigma)$ for each $\sigma \in \hat{G}$, which implies $T(\xi)f = f$ for all $\xi > 0$. The proof of (*a*) of the theorem will be complete if we show that an $f \in X$ belongs to the closed linear extension of J iff for each $\sigma \in \hat{G}, e^{\xi\nu(\sigma)}\hat{f}(\sigma)$ is independent of ξ. So, suppose that f is in the closed linear extension of J; then there is a sequence $\{f_n\}$ in $\mathscr{L}_J$ such that $\|f_n - f\| \to 0$. Let $\sigma \in \hat{G}$. If $\sigma \in J$, then $\nu(\sigma) = 0$ and hence $e^{\xi\nu(\sigma)}\hat{f}(\sigma) = \hat{f}(\sigma)$ is independent of ξ. If $\sigma \notin J$, then $\hat{f}_n(\sigma) = 0$ for each n. In this case $|\hat{f}(\sigma)| \leq |\hat{f}(\sigma) - \hat{f}_n(\sigma)| + |\hat{f}_n(\sigma)| \leq$ $\leq \|f - f_n\|$ implies that $|\hat{f}(\sigma)| = 0$. It follows that $\hat{f}(\sigma) = 0$ for each $\sigma \notin J$ and $e^{\xi\nu(\sigma)}\hat{f}(\sigma) = 0$ is again indepent of ξ. Conversely suppose $f \in X$ is such that, for each $\sigma \in \hat{G}$, $e^{\xi\nu(\sigma)}\hat{f}(\sigma)$ is independent of ξ. If $\sigma \notin J$, then $\nu(\sigma) \neq 0$, so we must have $\hat{f}(\sigma) = 0$. By the remarks following Definition 3.1, f must then belong to the closed linear extension of J.

(*b*) Let A_0 be the infinitesimal operator of $\mathscr{I} = \{T(\xi) : \xi > 0\}$. Since $\mathscr{I}$ is of class $(1, C_1)$, Theorem 10.7.2 of [1] implies that for each $f \in D(A_0)$, we have $T(\xi)f - f = \xi(A_0 f + o(1))$, for all $\xi > 0$. But since $\mathscr{I}$ is of class $(1, C_1)$, A_0 is closed ([1], Theorem 10.5.3), hence $A_0 = A$, the infinitesimal generator of $\mathscr{I}$. It follows that if $f \in D(A)$, then $T(\xi)f - f = \xi(Af + o(1))$ for all $\xi > 0$.

(*c*) The fact that $\mathscr{I}$ is of class $(1, C_1)$ implies that $\mathscr{I}$ is of class (A) ([1], Theorem 10.6.1). It now follows from Theorem 1.2 of [5] that $D(A) =$ $= \{f \in X : \nu\hat{f} = \hat{g}$ for some $g \in X\}$, i.e. ν is a $(D(A), X)$-multiplier.

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