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A note on certain generating functions for the classical polynomials

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Funzioni speciali. — *A note on certain generating functions for the classical polynomials* (*). Nota di HARI M. SRIVASTAVA, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore prova che alcuni risultati sulle funzioni generatrici ottenuti da O. Shanker [« J. Australian Math. Soc. », *15* (1973), 389–392] sono equivalenti ad altri noti risultati.

Si provano alcune generalizzazioni e la loro applicazione a famiglie polinomiali di Bessel, Brafman, Gegenbauer, Hermite, Jacobi e Laguerre.

1. INTRODUCTION

Given a sequence $\{\phi_n\}$ $(n \geqq 0)$, we define the new one $\{\psi_n\}$ by

$$\psi_n = \sum_{k=0}^{n} \binom{\alpha + \beta n}{n - k} \phi_k , \qquad \forall n \in \{0, 1, 2, \cdots\}, \tag{1}$$

where α and β are complex parameters independent of n.

Recently, O. Shanker [9] gave a class of generating functions and showed how his results would apply to the classical polynomials of Gegenbauer, Jacobi and Laguerre. {Unfortunately, however, his paper contains a number of errors and omissions, typographical or otherwise.} We recall here his main result [9, p. 389, Equation (2)] in the *corrected* form:

$$\sum_{n=0}^{\infty} \frac{\alpha (p + qn)}{\alpha + \beta n} \psi_n \left[\frac{x}{(1 + x)^{\beta}}\right]^n = \alpha (1 + x)^{\alpha} \sum_{n=0}^{\infty} \left[\frac{p + qn}{\alpha + \beta n} + \frac{qx}{1 + (1 - \beta) x}\right] \phi_n x^n , \tag{2}$$

where α, β, p and q are arbitrary complex numbers independent of n.

Formula (2) was given as a generalization of the following generating functions due to J. W. Brown [2]:

$$\sum_{n=0}^{\infty} \psi_n \left[\frac{x}{(1 + x)^{\beta}}\right]^n = \frac{(1 + x)^{\alpha+1}}{1 + (1 - \beta) x} \sum_{n=0}^{\infty} \phi_n x^n \tag{3}$$

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and

$$\text{(4)} \qquad \sum_{n=0}^{\infty} \frac{\alpha}{\alpha+\beta n}\, \psi_n \left[\frac{x}{(1+x)^{\beta}}\right]^n = (1+x)^{\alpha} \sum_{n=0}^{\infty} \frac{\alpha}{\alpha+\beta n}\, \phi_n\, x^n ,$$

where the sequences $\{\phi_n\}$ and $\{\psi_n\}$ are connected by means of (1). Indeed, as remarked by Shanker [9, p. 390], the generating function (3) would follow from (2) in the special case $p=1$, $q=\beta/\alpha$, and (2) with $p=1$, $q=0$ is the same as the generating function (4).

In his long and involved derivation of the generating function (2), Shanker (cf. [9], pp. 389-390) uses a known combinatorial identity in the recent book by J. Riordan [8, p. 169, Problem (16b)]. The object of the present note is first to show how readily the corrected version of Shanker's main result (and, of course, of each of its applications to the classical polynomials considered) would follow from Brown's generating functions (3) and (4) above. We then discuss certain relevant generalizations and indicate their possible applications not only to the classical Gegenbauer (or ultraspherical), Jacobi and Laguerre polynomials, but also to the Bessel polynomials of H. L. Krall and O. Frink [6, p. 108, Equation (34)], and to the Hermite polynomials and their generalizations by F. Brafman [1, p. 186, Equation (52)], and by H. W. Gould and A. T. Hopper [5, p. 58, Equation (6.2)].

2. EQUIVALENCE OF (2), (3) AND (4)

At the outset we must remark that the pair of generating functions (3) and (4) are not independent; in fact, they are equivalent in the sense that either one implies the other. For the proof of this interesting equivalence, the reader is referred to § 4 (pp. 409–410) of a recent paper by D. Zeitlin [12], who also gave a non-trivial generalization of (3) and (4). {See [12], p. 407, Theorem 2.}

Now we turn to Shanker's generalization (2) above. It is fairly straightforward to deduce (2) from the (Brown's) *equivalent* generating functions (3) and (4) by merely using elementary algebraic manipulations. As a matter of fact, if we multiply (3) and (4) by $\alpha q/\beta$ and $p-\alpha q/\beta$, respectively, and add the resulting equations, we shall at once arrive at the generating function (2). This evidently shows that the generating functions (3) and (4), together, imply the generating function (2).

Conversely, the generating function (2) can be shown to imply the generating functions (3) and (4). Indeed, the first member of (2) can be rewritten as

$$\text{(5)} \qquad \sum_{n=0}^{\infty} \frac{\alpha(p+qn)}{\alpha+\beta n}\, \psi_n \left[\frac{x}{(1+x)^{\beta}}\right]^n = \left(p-\frac{\alpha q}{\beta}\right) \sum_{n=0}^{\infty} \frac{\alpha}{\alpha+\beta n}\, \psi_n \left[\frac{x}{(1+x)^{\beta}}\right]^n + \frac{\alpha q}{\beta} \sum_{n=0}^{\infty} \psi_n \left[\frac{x}{(1+x)^{\beta}}\right]^n .$$

Now use the relationship (1) to substitute for ψ_n into each sum on the right-hand side of (5), change the order of the resulting double summations, and then apply one or the other of the following well-known consequences of Lagrange's expansion theorem (cf. Pólya and Szegö [7], p. 349, Problem 216 and p. 348, Problem 212; see also Gould [4], p. 86 *et seq.*):

$$\sum_{n=0}^{\infty} \binom{\alpha+\beta n}{n} \left[\frac{x}{(1+x)^{\beta}}\right]^n = \frac{(1+x)^{\alpha+1}}{1+(1-\beta)x} \tag{6}$$

and

$$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha+\beta n} \binom{\alpha+\beta n}{n} \left[\frac{x}{(1+x)^{\beta}}\right]^n = (1+x)^{\alpha}. \tag{7}$$

A comparison between the second member of (5), thus simplified, and the right-hand side of (2) will then yield the generating functions (3) and (4).

Since the generating functions (3) and (4) are equivalent, as observed earlier by Zeitlin [12, p. 406, Lemma 2], we have thus proved

THEOREM 1. *A necessary and sufficient condition for the generating function* (2) *to exist is* (3) *or* (4). {*More precisely, the generating functions* (2), (3), *and* (4) *are equivalent.*}

We remark in passing that the last identities (6) and (7) are not independent either. The proof of the fact that (6) implies (7) appeared in the aforecited book by Riordan (cf. [7], p. 148); on the other hand, L. Carlitz [2, p. 825] showed that (7) implies (6). {See also [12], p. 406, Lemma 1.}

3. GENERALIZATIONS

The various generating functions for the classical polynomials of Gegenbauer, Jacobi and Laguerre, given in § 3 of Shanker's paper [9], are immediate consequences of his main result (2) above; obviously, therefore, these are derivable also from Brown's equivalent generating functions (3) and (4). In this section we make use of Theorem 1 to derive a non-trivial generalization of the corrected version of the basic generating function in § 3 of Shanker's paper [9, p. 390, Equation (5)]. Our generating function (12) below would evidently apply to a larger variety of polynomial systems including, for instance, the Bessel polynomials (see Krall and Frink [6, p. 108, Equation (34)])

$$\begin{aligned} y_n(x,\alpha,\beta) &= {}_2F_0[-n, \alpha+n-1; -; -x/\beta] \\ &= n!\,(-x/\beta)^n L_n^{(1-\alpha-2n)}(\beta/x), \end{aligned} \tag{8}$$

the Hermite polynomials and their several familiar generalizations (see, for example, Brafman [1, p. 186, Equation (52)]; see also Gould and Hopper [5, p. 58, Equation (6.2)]).

We now state our main generating function given by

THEOREM 2. *Corresponding to every power series*

$$G(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad c_0 \neq 0, \tag{9}$$

let

$$S_n^{(\alpha,\beta)}(x;m) = \sum_{k=0}^{[n/m]} \binom{\alpha+\beta n}{n-mk} c_k x^k, \tag{10}$$

where α *and* β *are arbitrary complex parameters independent of* n, *and* m *is an arbitrary positive integer. Also let* v *be a function of* t *defined by*

$$v = t(1+v)^{\beta}, \qquad v(0) = 0. \tag{11}$$

Then

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{p+qn}{\alpha+\beta n} S_n^{(\alpha,\beta)}(x;m)\, t^n \\ &= (1+v)^{\alpha} \left\{ \sum_{n=0}^{\infty} \frac{p+qmn}{\alpha+\beta mn} c_n x^n v^{mn} + \frac{qv}{1+(1-\beta)v} G(xv^m) \right\}, \end{aligned} \tag{12}$$

provided that each side of (12) *has a meaning.*

Remark 1. A systematic study of the various systems of polynomials in one or more variables, related to those defined by (10), appeared in a number of recent papers including, for example, those by Srivastava [10], Srivastava and Buschman [11], and Zeitlin [12].

Remark 2. For $m = 1$, and with c_n replaced by $c_n/n!$, Theorem 2 will obviously provide us with the corrected versions of the basic equations (4) and (5) in § 3 of Shanker's paper (cf. [9], p. 390). {Notice that the factor α on each side of Shanker's result [9, p. 390, Equation (5)], as also in the generating function (2) above, is superfluous.}

Remark 3. In view of the definitions (see Brafman [1, p. 186, Equation (52)] and Gould and Hopper [5, p. 58, Equation (6.2)]), our general result (12) can be applied fairly easily to derive a class of generating functions for the Brafman and Gould-Hopper generalizations of the classical Hermite polynomials, which are all evidently contained in the generalized hypergeometric polynomials (see Srivastava [10, p. 233, Equation (12)])

$$H_n^{(\alpha,\beta)}(x;m) = \binom{\alpha+\beta n}{n} {}_{r+m}F_{s+m}\left[\begin{matrix} \Delta(m;-n),\, a_1,\cdots,a_r; \\ \Delta(m;1+\alpha+(\beta-1)n),\, b_1,\cdots,b_s; \end{matrix}\; x\right], \tag{13}$$

where, for convenience, $\Delta(m;\lambda)$ abbreviates the set of m parameters $(\lambda+j-1)/m$, $j = 1,\cdots,m$, $m \geq 1$, the set $\Delta(0;\lambda)$ being assumed to be empty.

The generalized hypergeometric polynomials in (13) would result from (10) if we set

$$(14) \qquad c_n = \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{(-1)^{mn}}{n!}, \qquad \forall n \in \{0, 1, 2, \cdots\},$$

where, as usual, $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)$.

Thus, as a consequence of Theorem 2, we obtain a class of hypergeometric generating functions given by

THEOREM 3. *Let α, β, p, q, and the parameters $a_i, i = 1, \cdots, r$, and $b_j, j = 1, \cdots, s$, be complex numbers independent of n such that $1 + \alpha/\beta m$, p/qm, and $b_j, j = 1, \cdots, s$, are neither zero nor a negative integer, m being any positive integer.*

Then the hypergeometric polynomials defined by (13) *are generated by*

$$(15) \qquad \sum_{n=0}^{\infty} \frac{p + qn}{\alpha + \beta n} \mathrm{H}_n^{(\alpha,\beta)}(x; m)\, t^n$$

$$= (1 + v)^{\alpha} \left\{ \frac{p}{\alpha}\, {}_{r+2}\mathrm{F}_{s+2} \left[\begin{matrix} \alpha/\beta m, 1 + p/qm, a_1, \cdots, a_r; \\ 1 + \alpha/\beta m, p/qm, b_1, \cdots, b_s; \end{matrix} \; x(-v)^m \right] \right.$$

$$\left. + \frac{qv}{1 + (1 - \beta) v}\, {}_r\mathrm{F}_s \left[\begin{matrix} a_1, \cdots, a_r; \\ b_1, \cdots, b_s; \end{matrix} \; x(-v)^m \right] \right\},$$

where v is given by (11).

We conclude by indicating that the generalized hypergeometric polynomials defined by (13) would correspond to the Bessel polynomials in (8) when $m = 1, \beta = -1$, and $r = s = 0$, to the classical Gegenbauer (or ultraspherical) polynomials when $m = 2, \beta = 1, r - 1 = s = 0$, and $a_1 = 1 + \alpha/2$, to certain special Jacobi (and, of course, Gegenbauer) polynomials when $m = 1$ and $r - 1 = s = 0$, to the Laguerre polynomials when $m = 1$ and $r = s = 0$, to the Hermite polynomials when $m = 2, \beta = 1$, $r - 2 = s = 0$, and $a_j = (\alpha + j)/2, j = 1, 2$, and to the aforementioned Gould-Hopper generalization of the classical Hermite polynomials when $\beta = 1, r - m = s = 0$, and $a_j = (\alpha + j)/m, \forall j \in \{1, \cdots, m\}$. And indeed, for $\beta = 1$, if we replace r by $r + m$, and set $a_{r+j} = (\alpha + j)/m, \forall j \in \{1, \cdots, m\}$, the hypergeometric polynomials defined by (13) would reduce essentially to the Brafman polynomials (cf. [1], p. 186, Equation (52)). Thus, in these special cases, our hypergeometric generating function (15) will readily apply to yield interesting generating functions for the various polynomial systems just enumerated, and for their particular forms available in the literature. The details involved are reasonably straightforward, and may well be left as an exercise to the interested reader.

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